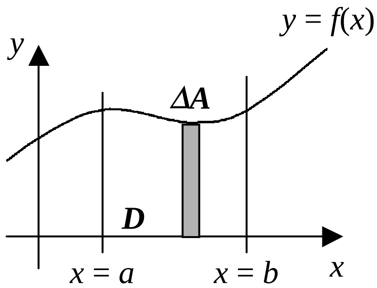
Introduction to Integral Calculus

(A Review of Chapter Five)



Let f be a continuous function with $f(x) \geq 0$ for $a \leq x \leq b$, and let $\boldsymbol{D}$ represent the region in the plane bounded by the x-axis, the vertical lines $x = a$, $x = b$, and the graph of $y = f(x)$. In symbols,

$$\boldsymbol{D} = \{0 \leq y \leq f(x), a \leq x \leq b\}.$$

Let A represent the area of $\boldsymbol{D}$. Then A may be approximated using tiny rectangles of area ΔA as illustrated. The areas of these rectangles are calculated by first forming a <u>partition</u> of the interval $[a,b]$ along the x-axis. This is done by choosing a set of points $\{x_k\}$, for $k = 1$ to N, with $a = x_0 < x_1 < x_2 < \ . \ . \ . \ < x_N = b$. This divides the interval $[a,b]$ into N subintervals $[x_{k-1},x_k]$ for $k = 1$ to N, each with length $\Delta x_k = x_k - x_{k-1}$ and forming the base of the rectangle A_k. The height of each rectangle is given by the function f, choosing $y_k = f(\xi_k)$ where ξ_k is any value of x in the subinterval $[x_{k-1},x_k]$.

A is thus approximated by summing the areas of these rectangles: $A \cong \sum_{k=1}^{N} A_k = \sum_{k=1}^{N} f(\xi_k)\Delta x_k$.

Taking the limit as N goes to infinity (and correspondingly as each Δx_k goes to zero) gives the definition of the <u>definite integral</u> of f over $[a,b]$ as

$$\int_a^b f(x)\,dx \ = \ \lim_{N \to \infty} \sum_{k=1}^{N} f(\xi_k)\Delta x_k \ .$$

Riemann proved this limit is always well defined (provided f is continuous or at least piecewise continuous over $[a,b]$), no matter how the partition is chosen. Thus, it makes sense to choose the partition which will make the calculation of the Riemann sums as easy as possible. Usually, the simplest way to do this is to use a <u>uniform</u> partition where each subinterval has the same length $\Delta x = (b - a)/N$ and each ξ_k is simply the right-hand endpoint of the subinterval $[x_{k-1},x_k]$. That is, $\xi_k = x_k = a + k\Delta x$ for $k = 1$ to N. This gives $A_k = f(x_k)\Delta x$, and summing these rectangles gives the Right-Endpoint Riemann Sum

$$R_N = \sum_{k=1}^{N} f(x_k)\Delta x \, .$$

If instead the left-hand endpoint of each subinterval is chosen, what is then formed is the Left-Endpoint Riemann Sum

$$L_N = \sum_{k=0}^{N-1} f(x_k)\Delta x \, ,$$

and taking the average of these two sums gives the result of the Trapezoidal Rule, for which we have theoretical upper bounds on the error in approximating a definite integral with a Riemann sum, in terms of bounds on the derivatives of f and the size of the partition. Your textbook also mentions the Midpoint Rule, taking $\xi_k = (x_{k-1} + x_k)/2$ as the midpoint of each subinterval. The Midpoint Rule usually gives a better approximation than the other sums, but it takes longer to calculate. Later (Chapter 7) we will discuss an even better approximation technique referred to as Simpson's Rule.

Motivation for Taking Calc II

If all that is needed is a way to calculate approximate values for definite integrals, then there is no reason to take Calc II, since Riemann Sums may be used to approximate definite integrals for continuous functions to as many decimal places as needed. In fact, many modern hand-held scientific calculators have such calculation devices already installed. However, Calculus is the Science of How to Calculate (without using a computer), and there are Tools of Integral Calculus which may be learned in order to make the computation of definite integrals easier, quite often easy enough to calculate by hand. It is the study of these topics, along with even more advanced techniques in approximation, which motivates a second semester course in calculus.

Tools of Integral Calculus

- Properties of the Definite Integral
- The Fundamental Theorem of Calculus
- Techniques of Integration
- Geometric Theorems
- Series Approximation Techniques

The properties of the definite integral and the Fundamental Theorem of Calculus are discussed in Chapter Five and should be covered at the end of a first-semester calculus course. One basic technique of integration, the Method of Substitution, is covered in §5.5, and other techniques will be discussed in Chapter Seven. Series methods are discussed in Chapter Eleven, and theorems from classical and analytic geometry will be discussed throughout the semester. We will also look at applications of integral calculus (Chapters Six, Eight, and Nine) and conclude the semester with some advanced topics from analytic geometry (Chapter Ten).

Properties of the Definite Integral (pages 373-375)

1. Area of Rectangle $\int_a^b c\,dx = c\cdot(b-a)$ (c is constant)
2. Linearity $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. Linearity $\int_a^b c\cdot f(x)\,dx = c\cdot\int_a^b f(x)\,dx$ (only if c is constant)
4. Linearity $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
5. Additivity $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
6. Comparison $f(x)\ge 0$ for x in $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge 0$
7. Comparison $f(x)\ge g(x)$ for x in $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$
8. Comparison $M_1 \le f(x) \le M_2 \Rightarrow M_1(b-a) \le \int_a^b f(x)\,dx \le M_2(b-a)$
9. Follows from 5 $\int_a^a f(x)\,dx = 0$
10. Follows from 5 $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

The Fundamental Theorem of Calculus

Let f be a continuous function with $f(x) \geq 0$ for $a \leq x \leq b$, and define the area function $A(t)$ for $a \leq t \leq b$ in terms of the definite integral: $A(t) = \int_a^t f(x)\,dx$

Note $A(a) = 0$ (see property 9 on the preceding page), and since f is nonnegative $A(t)$ must be an increasing function (the area under the curve increases as t increases). We know from Calc I that if a differentiable function is increasing then its derivative must be positive. To determine whether or not the area function is differentiable, we need to calculate the difference quotient: $\dfrac{\Delta A}{\Delta t} = \dfrac{A(t+\Delta t) - A(t)}{\Delta t}$

ΔA is calculated using the additivity of the definite integral: $\Delta A = \int_t^{t+\Delta t} f(x)\,dx$

Note ΔA uses values of $f(x)$ for x between t and $t + \Delta t$, so taking the limit as Δt goes to zero forces x to be equal to t. Thus, for Δt small enough, $f(x)$ can be replaced by $f(t)$, and so by property 1 of the definite integral $\Delta A \;\cong\; \int_t^{t+\Delta t} f(t)\,dx \;=\; \left[(t+\Delta t) - t\right] \cdot f(t) = f(t) \cdot \Delta t$, which means $\dfrac{\Delta A}{\Delta t}$ is approximately equal to $f(t)$ when Δt is small enough, and since f is continuous this approximation gets better as Δt gets smaller. Thus, taking the limit as Δt goes to zero shows the area function is differentiable, with its derivative given by $\dfrac{dA}{dt} = f(t)$. This is the statement of the Fundamental Theorem of Calculus.

In other words, if a function is constructed in terms of the definite integral for a continuous integrand f, then the constructed function is differentiable and its derivative is simply equal to f. Another way to state this is that the operations of integration and differentiation are inverse operations, i.e. they cancel each other out. This can be written symbolically as

$$\frac{d}{dt} \int_a^t f(x)\,dx = f(t),$$

which is sometimes referred to as Version One of the Fundamental Theorem of Calculus. Another way to say the operations of calculating integrals and derivatives are inverse operations is to state that integration can be performed using antiderivatives, where "F is an antiderivative of f" is true whenever f is the derivative of F. Letting $F(t) = \int_0^t f(x)\,dx$ gives F as an antiderivative of f, since the Fundamental Theorem of Calculus states the derivative of F is equal to f. Using the additivity of the definite integral gives

$$\int_a^b f(x)\,dx \;=\; \int_0^b f(x)\,dx - \int_0^a f(x)\,dx = F(b) - F(a),$$

which is the statement of Version Two of the Fundamental Theorem of Calculus. Both versions are given on page 387 of the text.

If the Fundamental Theorem of Calculus is used to calculate definite integrals, then the techniques of integration basically involve methods for finding antiderivatives. This is not as easy as the problem of calculating derivatives, since the definition of the derivative (in terms of the limiting value of a difference quotient) always gives a way to calculate the derivative, but the definition of the definite integral (in terms of the limiting value of a Riemann Sum) does not give any way to calculate antiderivatives. Thus the only way to calculate antiderivatives is to study the rules for derivatives and then figure out how to write those rules backwards.