A Contraction Mapping Proof of the Smooth Dependence on Parameters of Solutions to Volterra Integral Equations

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Abstract

We consider the linear Volterra equation

$$x(t) = a(t) - \int_0^t K(t,s)\,x(s)\,ds$$

and suppose that the kernel K and forcing function a depend on some parameters $\epsilon \in \mathbb{R}^d$. We prove that, under suitable conditions, the solutions depend on ϵ as smoothly the functions a and K. The proof is based on the *contraction mapping principle* and the *variational equation*. Though our conditions are not the most general possible they nonetheless include many important examples.

Key words: integral equation, Volterra, parameter dependence
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1. Introduction

We consider linear Volterra equations of the form

$$x(t) = a(t) - \int_0^t K(t,s)\,x(s)\,ds.$$

There are three basic questions to be addressed when confronted with such an equation: does a solution exist, is that solution unique, and is the solution stable?

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There are several approaches to proving the existence of solutions to differential and integral equations. Most fall into the categories of contraction mapping arguments, compactness arguments, or index theory arguments.

Of these the contraction mapping approach is probably the most elementary and has several advantages; the contraction mapping principle automatically gives uniqueness of the solution in some class and typically gives continuous dependence of the solution on the defining data.

There are many different forms of stability that have been defined for differential equations. Perhaps the weakest is continuous dependence of the solution on the functions a and K that define the equation. This has been widely studied, see for example Miller [Mil71] or Kelley [Kel73]. The work of Artstein gives a fairly complete picture of the problem of continuous dependence of solutions to Volterra integral equations [Art75a] [Art75b]. Continuous dependence can be thought of as weak stability under misspecification of the model.

However, in many problems arising out of physics the model is certain and any uncertainty in the model specification comes from uncertainty in the measurement of various physical parameters describing the model. Often this reduces our uncertainty to some finite dimensional vector space of parameters. In this case it is the natural to consider differentiable dependence of the solution on this finite dimensional space of parameters.

We prove such differential dependence on parameters in Theorem 8. A version of our theorem can be found in Gripenberg et al. [GLS90, Theorem 1.2 Chapter 13]. It comes as a corollary of a stronger theorem proved using compactness arguments.

2. Existence and Uniqueness of Solutions

We begin here by giving a version of the existence and uniqueness of solutions to the linear Volterra integral equation

$$x(t) = a(t) - \int_0^t K(t,s)x(s)\,ds. \tag{1}$$

As with differential equations an existence and uniqueness theorem can be obtained using the contraction mapping principle.

Theorem 1 (Contraction Mapping Principle)**.** *Let (X,d) be a complete metric space and $P : X \to X$ be a contraction mapping, i.e. there exists $0 \le \lambda < 1$ such that*

$$d\big(P(x), P(y)\big) \le \lambda d(x,y).$$

Then P has a unique fixed point, which we denote by $\omega(P)$. Moreover, for any $x \in X$ we have

$$d\big(x, \omega(P)\big) < \frac{d\big(x, P(x)\big)}{1-\lambda}.$$

First we wish to prove an existence and uniqueness theorem for finite time horizons. We denote the space of continuous functions with values in the normed space $(V, \|\cdot\|)$ on a compact metric space X by $C(X, V)$. We endow this space with the norm $\|\phi\|_0 = \sup_{x\in X} \|\phi(x)\|$ and write d_0 for the associated metric. We will use $V = \mathbb{R}^n$ endowed with the usual Euclidean norm and $V = M_n(\mathbb{R})$, the space of $n \times n$ matrices with entries in $\mathbb{R}$, endowed with the standard operator norm. For an interval $I \subseteq \mathbb{R}$ denote $\triangle(I) := \big\{(t, s) : s, t \in I, s \le t\big\}$.

Theorem 2 (Existence and Uniqueness for a Finite Time Horizon)**.** *Fix $T > 0$. Suppose $a \in C\big([0, T], \mathbb{R}^n\big)$ and $K : \triangle([0, T]) \to M_n(\mathbb{R})$ such that for every $\phi \in C\big([0, T], \mathbb{R}^n\big)$*

$$\int_0^t K(t, s)\phi(s)\, ds \tag{2}$$

is a continuous function of t and there exists $\lambda < 1$ and such that

$$\sup_{t\in[0,T]} \int_0^t \|K(t, s)\|\, ds < \lambda. \tag{3}$$

Then there exists a unique bounded solution x to (1) and $x \in C\big([0, T], \mathbb{R}^n\big)$.

Remark 1. If we have $K \in C\big(\triangle([0, T]), M_n(\mathbb{R})\big)$ then our condition (2) holds. However our kernel

$$K(t, s) = \begin{cases} \frac{1}{1+(t-s)} & t - 1 \le s \le t \\ 0 & s < t - 1 \end{cases}$$

satisfies (2) even though it is not continuous.

Proof. Define the map $P : C\big([0, T], \mathbb{R}^n\big) \to C\big([0, T], \mathbb{R}^n\big)$ by

$$P(\phi)(t) := a(t) - \int_0^t K(t, s)\phi(s)\, ds. \tag{4}$$

We have

$$\begin{aligned} d_0\big(P(\phi_1), P(\phi_2)\big) &\le \sup_{t\in[0,T]} \Big\| \int_0^t K(t, s)(\phi_1(s) - \phi_2(s))\, ds \Big\| \\ &\le \sup_{t\in[0,T]} \int_0^t \|K(t, s)\| \|\phi_1(s) - \phi_2(s)\|\, ds \\ &\le \sup_{t\in[0,T]} \int_0^t \|K(t, s)\|\, ds \sup_{t\in[0,T]} \|\phi_1(t) - \phi_2(t)\| \\ &< \lambda\, d_0\big(\phi_1, \phi_2\big) \end{aligned}$$

where λ is from (3). Thus P is a contraction mapping on $C\big([0, T], \mathbb{R}^n\big)$. Thus we get a unique fixed point $x(t)$ in $C\big([0, T], \mathbb{R}^n\big)$. If we apply our estimate from the contraction mapping principle to the function 0 we get

$$\|x\|_0 = d_0(0, x) \le \frac{d_0\big(0, P(0)\big)}{1 - \lambda} = \frac{\|a\|_0}{1 - \lambda}.$$

It remains to prove that the solution is unique. Suppose that x_1 and x_2 are two bounded solutions to the equation (1) then consider $\Delta(t) = x_1(t) - x_2(t)$. This satisfies the integral equation

$$\Delta(t) = -\int_0^t K(t,s)\Delta(s)\,ds$$

and hence we have the estimate

$$|\Delta(t)| \le \int_0^t \|K(t,s)\|\,|\Delta(s)|\,ds$$

Suppose that $\Delta_* = \sup_{[0,T]} |\Delta(t)| > 0$. Choose $t_* \in [0,T]$ such that $|\Delta(t_*)| > \lambda\,\Delta_*$. Now

$$\begin{aligned} |\Delta(t_*)| &\le \int_0^{t_*} \|K(t_*,s)\|\,|\Delta(s)|\,ds \\ &\le \int_0^{t_*} \|K(t_*,s)\|\,ds\;\Delta_* \\ &\le \lambda\,\Delta_* \end{aligned}$$

which is a contradiction. Hence $\Delta(t) \equiv 0$ and $x_1(t) = x_2(t)$. □

Remark 2. One can extend the class of kernels that can be treated using the contraction mapping principle considerably by introducing exponentially weighted norms. We replace our norm on $C\big([0,T],\mathbb{R}^n\big)$ by a family of equivalent exponentially weighted norms

$$\|\phi\|_r := \sup_{t\in[0,T]} e^{-rt}\|\phi(t)\| \tag{5}$$

for $r > 0$. The we can replace our contraction condition (3) by the much weaker condition that

$$\kappa := \sup_{t\in[0,T]} \int_0^t \|K(t,s)\|^p\,ds < \infty$$

for some $p > 1$. In this case one obtains an estimate

$$d_r\big(P(\phi_1),P(\phi_2)\big) \le \frac{\kappa}{(q\,r)^{1/q}}\,d_r\big(\phi_1,\phi_2\big)$$

where q is the Hölder conjugate of p. For a suitable choice of r we recover a contraction and proceed as above. As we aim to make the exposition as elementary as possible we choose not to pursue this.

Now we give two slightly different versions of the existence and uniqueness theorem for the infinite time horizon. We denote by $C_b(X,\mathbb{R})$ the space of bounded continuous real-valued functions on X endowed with the supremum metric.

Theorem 3 (Existence and Uniqueness for Infinite Time Horizon 1)**.** *Suppose* $a \in C_b\big([0,\infty),\mathbb{R}^n\big)$ *and* $K : \triangle([0,\infty)) \to M_n(\mathbb{R})$ *such that*

1. *for every* $\phi \in C\big([0,\infty),\mathbb{R}^n\big)$

$$\int_0^t K(t,s)\phi(s)\,ds$$

   is a continuous function of t*.*
2. $\lambda := \sup_{t\geq 0}\int_0^t \|K(t,s)\|\,ds < 1$.

Then there is a unique solution x *to* (1) *and* $x \in C_b\big([0,\infty),\mathbb{R}^n\big)$*. Moreover*

$$\|x\|_0 \leq \frac{\|a\|_0}{1-\lambda}.$$

We can either repeat the contraction mapping proof from Theorem 2 above or we can observe that we get a unique continuous solution x on every interval $[0,T]$ which satisfies the required bound and hence by extension we have a continuous solution x on $[0,\infty)$ which satisfies the required bound. If we replace our condition 1 with the assumption that $K \in C\big(\triangle([0,\infty)), M_n(\mathbb{R})\big)$ then is the result [Bur08, Theorem 0.2.1].

Theorem 4 (Existence and Uniqueness for Infinite Time Horizon 2)**.** *Suppose* $a \in C\big([0,\infty),\mathbb{R}^n\big)$*,* $K : \triangle([0,\infty)) \to M_n(\mathbb{R})$ *, and that*

1. *for every* $\phi \in C\big([0,\infty),\mathbb{R}^n\big)$

$$\int_0^t K(t,s)\phi(s)\,ds$$

   is a continuous function of t*,*
2. *for every* $T > 0$

$$\sup_{t\in[0,T]}\int_0^t \|K(t,s)\|\,ds < 1.$$

Then there is a unique solution x *to* (1) *and* $x \in C\big([0,\infty),\mathbb{R}^n\big)$*.*

Here we just observe that we get a unique continuous solution x on every interval $[0,T]$ and hence by the extension we have a continuous solution x on $[0,\infty)$. The solution need not be bounded. This version of the result works for $K(t,s) = e^{-(t-s)}$ what has $\sup_{t\geq 0}\int_0^t |K(t,s)|\,ds = 1$.

3. Continuous Dependence

Next we wish to see how the solution x of (1) depends on the function a and the kernel K. This shows stability properties of the solution with respect to model specification. These results are useful when the model is constructed from data.

Our proof of existence and uniqueness used the contraction mapping principle. Therefore we begin by examining how the fixed point of a contraction mapping depends on the mapping.

We will denote the space of contraction maps on (X, d) by $\mathrm{Ctr}(X)$. We will consider it as a subspace of the continuous functions on X, $C(X)$, endowed with the supremum metric, denoted by d_0.Notice that $\mathrm{Ctr}(X)$ is not an open subset of $C(X)$. The space $\mathrm{Ctr}(X)$ is an open subspace of the space of Lipshitz functions endowed with the Lipshitz metric but we will not use that here.

Lemma 1. *The function $\omega : \mathrm{Ctr}(X) \to X$ is continuous.*

Proof. Let $P \in \mathrm{Ctr}(X)$. By definition we have

$$d\big(P(x), P(y)\big) \leq \lambda\, d(x, y).$$

for some $0 < \lambda < 1$. Let $\epsilon > 0$ be arbitrary and define $\delta = (1 - \lambda)\epsilon$. Suppose $Q \in \mathrm{Ctr}(X)$ with $d_0(P, Q) < \delta$. We apply the estimate from the contraction mapping principle for P to $\omega(Q)$

$$\begin{aligned} d\big(\omega(Q), \omega(P)\big) &< \frac{d\big(\omega(Q), P(\omega(Q))\big)}{1 - \lambda} \\ &= \frac{d\big(Q(\omega(Q)), P(\omega(Q))\big)}{1 - \lambda} < \frac{\delta}{1 - \lambda} = \epsilon. \end{aligned}$$

Thus we have that ω is a continuous function. $\square$

We will begin with the case of a finite time horizon again.

Theorem 5 (Continuous Dependence for a Finite Horizon)**.** *Fix $T > 0$. Define*

$$\mathscr{K} = \Big\{K \in C\big(\triangle([0, T]), M_n(\mathbb{R})\big) : \sup_{t\in[0,T]} \int_0^t \|K(t, s)\|\, ds < 1\Big\}$$

For every $a \in C\big([0, T], \mathbb{R}^n\big)$ and $K \in \mathscr{K}$ we get a unique solution $x_{a,K} \in C\big([0, T], R^n\big)$. The map

$$X : C\big([0, T], \mathbb{R}^n\big) \times \mathscr{K} \to C\big([0, T], R^n\big),$$

given by $X(a, K) = x_{a,K}$, is continuous.

Proof. We fix $a \in C\big([0, T], \mathbb{R}^n\big)$ and $K \in \mathscr{K}$. Consider $\mathscr{B} = \{\phi \in C\big([0, T], R^n\big) : d_0(\phi, x_{a,K}) < 1\}$. Observe that the mapping

$$P_{a,K}(\phi)(t) = a(t) - \int_0^t K(t, s)\phi(s)\, ds$$

induces a contraction on $\mathscr{B}$ with contraction factor

$$\lambda = \sup_{t\in[0,T]} \int_0^t \|K(t, s)\|\, ds < 1.$$

Suppose $a' \in C\big([0,T],\mathbb{R}^n\big)$ and $K' \in \mathscr{K}$. Observe that

$$\|P_{a',K'}(\phi) - P_{a,K}(\phi)\|_0 \le \|a'-a\|_0 + T\,\|K'-K\|_0\,\|\phi\|_0 \tag{6}$$

and hence if

$$\|a'-a\|_0 + T\,\|K'-K\|_0\,(\|x_{a,K}\|_0 + 1) < 1-\lambda$$

then $P_{a',K'}$ induces a contraction mapping on $\mathscr{B}$. Let $\mathscr{P}$ be an open set of functions (a',K') containing (a,K) such that $P_{a',K'}$ induces a contraction mapping on $\mathscr{B}$,

$$\mathscr{P} = \{(a',K') : \|a'-a\|_0 + T\|K'-K\|_0(\|x_{a,K}\|_0+1) < 1-\lambda\}.$$

The same condition (6) shows that the map $P : \mathscr{P} \to \mathrm{Ctr}(\mathscr{B})$ given by $P(a',K') = P_{a',K'}$ is continuous at (a,K). By our earlier lemma the map from $\mathrm{Ctr}(\mathscr{B})$ to the solution is continuous. Hence the map from $\mathscr{P}$ to the solution is continuous at (a,K). Since $\mathscr{P}$ is an open subset of $C\big([0,T],\mathbb{R}^n\big) \times \mathscr{K}$ we see that X is continuous at (a,K). $\square$

In fact the result gives some quantitative estimates. Suppose that

$$\lambda = \sup_{t\in[0,T]} \int_0^t \|K(t,s)\|\,ds < 1.$$

and that for $a' \in C\big([0,T],\mathbb{R}^n\big)$ and $K' \in \mathscr{K}$ we have $m \in \mathbb{R}$ such that

$$\|a'-a\|_0 + T\|K'-K\|_0(\|x_{a,K}\|_0 + m) < (1-\lambda)m$$

then

$$d_0(x_{a,K}, x_{a',K'}) < \frac{\|a'-a\|_0}{1-\lambda} + \frac{T\|K'-K\|_0(\frac{\|a\|_0}{1-\lambda}+m)}{1-\lambda}.$$

Before proceeding to an infinite horizon case we need to modify our estimate slightly. Observe that we used the relatively brutal estimate

$$\sup_{t\in[0,T]} \int_0^t \|K(t,s)-K'(t,s)\|\,ds \le T\|K(t,s)-K'(t,s)\|_0.$$

If we instead define

$$d_1\big(K(t,s),K'(t,s)\big) := \sup_{t\in[0,T]} \int_0^t \|K(t,s)-K'(t,s)\|\,ds \tag{7}$$

then we obtain the estimate

$$d_0(x_{a,K}, x_{a',K'}) < \frac{\|a'-a\|_0}{1-\lambda} + \frac{d_1\big(K(t,s),K'(t,s)\big)(\frac{\|a\|_0}{1-\lambda}+m)}{1-\lambda}.$$

The finite horizon result we stated immediately yields an infinite horizon result.

Theorem 6 (Continuous Dependence for the Infinite Horizon 1). *Define*

$$\mathscr{K} = \Big\{K \in C\big(\triangle\big([0,\infty)\big), M_n(\mathbb{R})\big) : \forall T > 0 \sup_{t\in[0,T]} \int_0^t \|K(t,s)\|\, ds < 1\Big\}$$

For every $a \in C\big([0,\infty), \mathbb{R}^n\big)$ and $K \in \mathscr{K}$ we get a unique solution $x_{a,K} \in C\big([0,\infty), R^n\big)$. The map

$$X : C\big([0,\infty), \mathbb{R}^n\big) \times \mathscr{K} \to C\big([0,\infty), R^n\big),$$

given by $X(a,K) = x_{a,K}$, is continuous where all three spaces of continuous functions are endowed with the topology of uniform convergence on compact sets.

However this gives no control over the solution at infinity. To get control over the solution at infinity we require stronger hypotheses.

Theorem 7 (Continuous Dependence for the Infinite Horizon 2). *Define*

$$\mathscr{K} = \Big\{K \in C\big(\triangle\big([0,\infty)\big), M_n(\mathbb{R})\big) : \sup_{t\in[0,\infty)} \int_0^t \|K(t,s)\|\, ds < 1\Big\}$$

For every $a \in C_b\big([0,\infty), \mathbb{R}^n\big)$ and $K \in \mathscr{K}$ we get a unique solution $x_{a,K} \in C_b\big([0,\infty), R^n\big)$. The map

$$X : C_b\big([0,\infty), \mathbb{R}^n\big) \times \mathscr{K} \to C_b\big([0,\infty), R^n\big),$$

given by $X(a,K) = x_{a,K}$, is continuous where $C_b\big([0,\infty), \mathbb{R}^n\big)$ is endowed with the supremum metric and $\mathscr{K}$ is endowed with the metric d_1 from (7)*.*

4. Smooth Dependence on Parameters

Our argument will be an inductive one. We begin with the crucial lemma that shows the existence of one derivative for an equation which depends on a single parameter ϵ.

Lemma 2. *Let $U \subseteq \mathbb{R}$ be open. Let*

$$\mathscr{K} = \Big\{K \in C\big(\triangle([0,T]), M_n(\mathbb{R})\big) : \sup_{t\in[0,T]} \int_0^t \|K(t,s)\|\, ds < 1\Big\}.$$

Suppose that $a \in C\big(U \times [0,T], \mathbb{R}^n\big)$ and $K \in C\big(U \times \triangle([0,T]), M_n(\mathbb{R})\big)$ with $\frac{\partial a}{\partial \epsilon} \in C\big(U \times [0,T], \mathbb{R}^n\big)$ and $\frac{\partial K}{\partial \epsilon} \in C\big(U \times \triangle([0,T]), M_n(\mathbb{R})\big)$. For each $\epsilon \in U$ we have a continuous solution x_ϵ to

$$x_\epsilon(t) = a_\epsilon(t) - \int_0^t K_\epsilon(t,s)\, x_e(s)\, ds.$$

The solution x_ϵ is differentiable in ϵ with $\frac{\partial x}{\partial \epsilon} \in C\big(U \times [0,T], \mathbb{R}^n\big)$. Moreover $\frac{\partial x}{\partial \epsilon}$ satisfies the Volterra integral equation

$$\frac{\partial x_\epsilon}{\partial \epsilon}(t) = \Big(\frac{\partial a_\epsilon}{\partial \epsilon}(t) - \int_0^t \frac{\partial K_\epsilon(t,s)}{\partial \epsilon} x(s)\, ds\Big) - \int_0^t K(t,s) \frac{\partial x}{\partial \epsilon}(s)\, ds$$

Proof. For each $\epsilon \in U$ we have

$$\sup_{t \in [0,T]} \int_0^t \|K_\epsilon(t,s)\|\, ds < 1$$

and $a_\epsilon \in C\big([0,T], \mathbb{R}\big)$. Hence by Theorem 2 we have a continuous solution $x_\epsilon(t)$ to (1). Futhermore, since $a_\epsilon \in C\big([0,T], \mathbb{R}^n\big)$ and $K_\epsilon \in C\big(\triangle([0,T]), M_n(\mathbb{R})\big)$ depend continuously on ϵ by Theorem 5 x_ϵ depends continuously on ϵ. We wish to show that x_ϵ is differentiable with respect to ϵ.

We consider formally differentiating the equation with respect to ϵ to obtain

$$\begin{aligned} \frac{\partial x_\epsilon}{\partial \epsilon}(t) &= \frac{\partial a_\epsilon}{\partial \epsilon} - \int_0^t \big(\frac{\partial K_\epsilon(t,s)}{\partial \epsilon} x_\epsilon(s) + K_\epsilon(t,s) \frac{\partial x_\epsilon}{\partial \epsilon}(s)\big)\, ds \\ &= \Big(\frac{\partial a}{\partial \epsilon} - \int_0^t \frac{\partial K(t,s)}{\partial \epsilon} x(s)\, ds\Big) - \int_0^t K_\epsilon(t,s) \frac{\partial x}{\partial \epsilon}(s)\, ds \end{aligned} \tag{8}$$

By hypothesis

$$\frac{\partial a_\epsilon}{\partial \epsilon} - \int_0^t \frac{\partial K_\epsilon(t,s)}{\partial \epsilon} x_\epsilon(s)\, ds$$

is continuous. Hence applying Theorem 2 we get a continuous solution to equation (8). We will denote solution by $\widehat{\frac{\partial x_\epsilon}{\partial \epsilon}}$.

Though $\widehat{\frac{\partial x_\epsilon}{\partial \epsilon}}$ satisfies the formal equation for the derivative we still need to show it is in fact the derivative. To do that we compare $\widehat{\frac{\partial x_\epsilon}{\partial \epsilon}}$ with the difference quotient

$$q_{\epsilon,\Delta\epsilon}(t) = \frac{x_{\epsilon+\Delta\epsilon}(t) - x_\epsilon(t)}{\Delta\epsilon}.$$

The difference quotient $q_{\epsilon,\Delta\epsilon}(t)$ also satisfies an integral equation

$$q_{\epsilon,\Delta\epsilon}(t) = \frac{a_{\epsilon+\Delta\epsilon}(t) - a_\epsilon(t)}{\Delta\epsilon} - \int_0^t \frac{K_{\epsilon+\Delta\epsilon}(t,s) x_{\epsilon+\Delta\epsilon}(s) - K_\epsilon(t,s) x_\epsilon(s)}{\Delta\epsilon}\, ds$$

Now

$$\begin{aligned} &\frac{K_{\epsilon+\Delta\epsilon}(t,s) x_{\epsilon+\Delta\epsilon}(s) - K_\epsilon(t,s) x_\epsilon(s)}{\Delta\epsilon} \\ &\quad= \frac{\big(K_{\epsilon+\Delta\epsilon}(t,s) - K_\epsilon(t,s)\big) x_\epsilon(s) + K_{\epsilon+\Delta\epsilon}(t,s)\big(x_{\epsilon+\Delta\epsilon}(s) - x_\epsilon(s)\big)}{\Delta\epsilon} \\ &\quad= \frac{K_{\epsilon+\Delta\epsilon}(t,s) - K_\epsilon(t,s)}{\Delta\epsilon} x_{\epsilon+\Delta\epsilon}(s) + K_\epsilon(t,s) \frac{x_{\epsilon+\Delta\epsilon}(s) - x_\epsilon(s)}{\Delta\epsilon} \\ &\quad= \frac{K_{\epsilon+\Delta\epsilon}(t,s) - K_\epsilon(t,s)}{\Delta\epsilon} x_{\epsilon+\Delta\epsilon}(s) + K_\epsilon(t,s) q_{\epsilon,\Delta\epsilon}(s) \end{aligned}$$

Finally we consider the equation satisfied by $\widehat{\frac{\partial x_\epsilon}{\partial \epsilon}}(t) - q_{\epsilon,\Delta\epsilon}(t)$

$$\begin{aligned}
\widehat{\frac{\partial x_\epsilon}{\partial \epsilon}}&(t) - q_{\epsilon,\Delta\epsilon}(t) \\
&= \frac{\partial a_\epsilon}{\partial \epsilon}(t) - \frac{a_{\epsilon+\Delta\epsilon}(t) - a_\epsilon(t)}{\Delta\epsilon} \\
&\quad - \int_0^t \Big(\frac{\partial K_\epsilon(t,s)}{\partial \epsilon} x_\epsilon(s) - \frac{K_{\epsilon+\Delta\epsilon}(t,s) - K_\epsilon(t,s)}{\Delta\epsilon} x_{\epsilon+\Delta\epsilon}(s)\Big)\, ds \\
&\qquad - \int_0^t K_\epsilon(t,s)\Big(\widehat{\frac{\partial x_\epsilon}{\partial \epsilon}}(s) - q_{\epsilon,\Delta\epsilon}(s)\Big)\, ds.
\end{aligned}$$

We have assumed that $\frac{\partial K_\epsilon}{\partial \epsilon} \in C\big(\triangle([0,T]), M_n(\mathbb{R})\big)$ depends continuously on ϵ and by Theorem 2 we have that $x_\epsilon \in C\big([0,T], \mathbb{R}^n\big)$ depends continuously on ϵ, so

$$\lim_{\Delta\epsilon\to 0} \Big\| \frac{\partial a_\epsilon}{\partial \epsilon}(t) - \frac{a_{\epsilon+\Delta\epsilon}(t) - a_\epsilon(t)}{\Delta\epsilon} \Big\|_0 = 0$$
$$\lim_{\Delta\epsilon\to 0} \Big\| \frac{\partial K_\epsilon(t,s)}{\partial \epsilon} x_\epsilon(s) - \frac{K_{\epsilon+\Delta\epsilon}(t,s) - K_\epsilon(t,s)}{\Delta\epsilon} x_{\epsilon+\Delta\epsilon}(s) \Big\|_0 = 0.$$

From the estimate in Theorem 2 we have

$$\lim_{\Delta\epsilon\to 0} \Big\| \widehat{\frac{\partial x_\epsilon}{\partial \epsilon}}(t) - q_{\epsilon,\Delta\epsilon}(t) \Big\| = 0$$

and consequently we see that our candidate is the true derivative. An application of Theorem 5 shows that $\frac{\partial x}{\partial \epsilon} \in C\big(U \times [0,T], \mathbb{R}^n\big)$.

□

Let $U \subseteq \mathbb{R}^d$. Given $(\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ define $|\alpha| = \alpha_1 + \cdots + \alpha_d$. We define

$$\frac{d^{|\alpha|}}{d\epsilon^\alpha} := \frac{d^{|\alpha|}}{\partial \epsilon_1^{\alpha_1} \ldots \partial \epsilon_d^{\alpha_d}}.$$

Now we use induction to extend this to a finite dimensional set of parameters and to higher derivatives.

Theorem 8 (Differentiable Dependence for a Finite Horizon)**.** *Let $U \subseteq \mathbb{R}^d$ be open. Let $r \in \mathbb{N}$. Let*

$$\mathscr{K} = \Big\{ K \in C\big(\triangle([0,T]), M_n(\mathbb{R})\big) : \sup_{t\in[0,T]} \int_0^t \|K(t,s)\|\, ds < 1 \Big\}.$$

Suppose that $a \in C\big(U \times [0,T], \mathbb{R}^n\big)$ and $K \in C\big(U \times \triangle([0,T]), M_n(\mathbb{R})\big)$ with for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \le r$

$$\frac{\partial^{|\alpha|} a}{\partial \epsilon^\alpha} \in C\big(U \times [0,T], \mathbb{R}^n\big)$$
$$\frac{\partial^{|\alpha|} K}{\partial \epsilon^\alpha} \in C\big(U \times \triangle([0,T]), M_n(\mathbb{R})\big)$$

For each $\epsilon \in U$ we have a continuous solution x_ϵ to

$$x_\epsilon(t) = a_\epsilon(t) - \int_0^t K_\epsilon(t,s)\, x_e(s)\, ds.$$

The solution x_ϵ is differentiable in ϵ with $\frac{\partial^{|\alpha|} x}{\partial \epsilon^\alpha} \in C\big(U \times [0,T], \mathbb{R}^n\big)$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \le r$.

Proof. We take as our inductive hypothesis that for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \le k < n$ the partial derivative $\frac{\partial^{|\alpha|} x}{\partial \epsilon^\alpha} \in C\big(U \times [0,T], \mathbb{R}^n\big)$ and satisfies a Volterra integral equation of the form

$$\frac{\partial^{|\alpha|} x_\epsilon}{\partial \epsilon^\alpha}(t) = a_\epsilon^{(\alpha)}(t) - \int_0^t K_\epsilon(t,s)\, \frac{\partial^{|\alpha|} x_\epsilon}{\partial \epsilon^\alpha}(s)\, ds.$$

We wish to differentiate with respect to ϵ_i for $1 \le i \le d$. We imagine fixing the other parameters and differentiate with respect to ϵ_i using Lemma 2. The new derivative satisfies the Volterra equation

$$\begin{aligned} &\frac{\partial}{\partial \epsilon_i} \frac{\partial^{|\alpha|} x_\epsilon}{\partial \epsilon^\alpha}(t) \\ &\quad = \Big(\frac{\partial a_\epsilon^{(\alpha)}}{\partial \epsilon_i}(t) - \int_0^t \frac{\partial K_\epsilon(t,s)}{\partial \epsilon_i}\, \frac{\partial^{|\alpha|} x_\epsilon}{\partial \epsilon^\alpha}(s)\, ds \Big) - \int_0^t K_\epsilon(t,s)\, \frac{\partial}{\partial \epsilon_i} \frac{\partial^{|\alpha|} x_\epsilon}{\partial \epsilon^\alpha}(s)\, ds. \end{aligned}$$

Our inductive hypothesis and assumptions ensure that

$$\frac{\partial a_\epsilon^{(\alpha)}}{\partial \epsilon_i}(t) - \int_0^t \frac{\partial K_\epsilon(t,s)}{\partial \epsilon_i}\, \frac{\partial^{|\alpha|} x_\epsilon}{\partial \epsilon^\alpha}(s)\, ds$$

is a continuous function of ϵ. We appeal to Theorem 5 to prove that

$$\frac{\partial}{\partial \epsilon_i} \frac{\partial^{|\alpha|} x}{\partial \epsilon^\alpha} \in C\big(U \times [0,T], \mathbb{R}^n\big).$$

Applying this to each i in turn we conclude that that for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \le k+1$ the partial derivative $\frac{\partial^{|\alpha|} x}{\partial \epsilon^\alpha} \in C\big(U \times [0,T], \mathbb{R}^n\big)$ and satisfies a Volterra integral equation of the form

$$\frac{\partial^{|\alpha|} x_\epsilon}{\partial \epsilon^\alpha}(t) = a_\epsilon^{(\alpha)}(t) - \int_0^t K_\epsilon(t,s)\, \frac{\partial^{|\alpha|} x_\epsilon}{\partial \epsilon^\alpha}(s)\, ds.$$

The induction continues until we can no longer differentiate a and K. ☐

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