Avoiding Early Closing: A Corrigendum to *"Livšic Theorems. . . "*†

Rafael de la Llave††, and Alistair Windsor‡

† *Department of Mathematics, University of Texas at Austin, 1 University Station, C1200, Austin, Texas 78712.*

‡ *373 Dunn Hall, Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152-3240.*

(*Received 27 November 2007*)

1. *Introduction*

This paper serves as a corrigendum to our paper [**dlLW09**]. In particular, in the proof of Theorem 6.3 we claimed the following:

CLAIM: *Let f be a topologically transitive Anosov diffeomorphism of a compact manifold M. For all $\epsilon > 0$ there exists $L \geq 1$ such that for every $n \in \mathbb{N}$ and $x \in M$ there exists a periodic point $p \in M$ satisfying*

1. *for all $0 \leq i \leq n$,*
$$d_M(f^i x, f^i p) < \epsilon,$$
and
2. *p has minimal period $n + \ell$ with $0 \leq \ell \leq L$.*

Unfortunately, as B. Kalinin and V. Sadovskaya discovered, the proof sketched contained gaps. Using specification as was suggested in our paper leads to a weaker result than we claimed. In this paper we prove a uniform version of closing :

THEOREM: *Let f be a topologically transitive C^1 Anosov diffeomorphism of a compact connected manifold M. Given $\epsilon > 0$ there exists $D \geq 1$ and $N > 0$ such that for all $x \in M$ and $n \in \mathbb{N}$ there exists a periodic point $p \in M$ with minimal period $m \in \mathbb{N}$ and $d \in \mathbb{N}$ such that:*

1. *for all $0 \leq i \leq n - 1$*
$$d_M(f^i x, f^i p) < \epsilon,$$
and

† RAFAEL DE LA LLAVE and ALISTAIR WINDSOR Livšic theorems for non-commutative groups including diffeomorphism groups and results on the existence of conformal structures for Anosov systems. *Ergodic Theory and Dynamical Systems*, Published online by Cambridge University Press 17 Jul 2009 doi:10.1017/S014338570900039X

† R. de la Llave was supported by the NSF grant NSF DMS 0354567.

2. $\quad n \leq d \cdot m \leq n + N$ *and* $1 \leq d \leq D$.

This result is strong enough to complete the proof of Theorem 6.3.

2. *Results*

To prove our result for Anosov diffeomorphisms we will first prove a similar statement for subshifts of finite type. Since every Anosov diffeomorphism is a factor of a subshift this will allow us to prove the desired result.

Recall that a subshift of finite type can be described by a transition matrix A. Symbol j may follow symbol i in a word in Σ_A if $A_{i,j} = 1$. A finite sequence $(a_1, \ldots, a_n)$ is called admissible if $A_{a_i, a_{i+1}} = 1$ for $0 \leq i \leq n-1$. We will call a finite sequence $(a_1, \ldots, a_n)$ periodic if it is admissible and $A_{a_n, a_1} = 1$ so that the sequence can be extended periodically to a point $a \in \Sigma_A$ of period n.

The following result is similar to that of Fine and Wilf [**FW65**].

LEMMA 1. *Let (Σ_A, σ) be a subshift of finite type. Let $(a_1, \ldots, a_n)$ be a periodic sequence of period m_1. Let $(a_1, \ldots, a_n, \ldots, a_{n+L})$ be an extension of $(a_1, \ldots, a_n)$ and be periodic with period m_2.*

If $m_1 + m_2 \leq n$ then $(a_1, \ldots, a_n)$ and $(a_1, \ldots, a_{n+L})$ are both periodic of period $\gcd(m_1, m_2)$.

Proof: Write $d := \gcd(m_1, m_2) = k_1 m_1 + k_2 m_2$ with $k_1, k_2 \in \mathbb{Z}$. Consider the following variation on the proof of Bézout's theorem that only uses numbers in the range $1, \ldots, n$. If $k_1 > 0$ then define $k_+ = k_1$, $m_+ = m_1$, $k_- = -k_2$ and $m_- = m_2$. If $k_2 > 0$ then define $k_+ = k_2$, $m_+ = m_2$, $k_- = -k_1$ and $m_- = m_1$

Let $1 \leq i \leq n - d$ be arbitrary and initialize $k = i$.

1. Add m_+ to k successively until either
   (a) adding a further m_+ would give k above n, or
   (b) all k_+ of the m_+s have been used.
2. Subtract m_- from the new k successively until either
   (a) subtracting a further m_- would give k below 1, or
   (b) all k_- of the m_-s have been used.
3. If $k \neq i + d$ then return to step 1.

Notice that if $k + m_+ \geq n + 1$ and $k - m_- \leq 0$ then $m_1 + m_2 \geq n + 1$ which is a contradiction. Thus this cannot terminate at an intermediate stage and the algorithm must proceed to give $k = i + d$.

Since each of these steps involves one of the two periods and all of the numbers are within $1, \ldots, n$ this shows that $a_i = a_{i+d}$ for $1 \leq i \leq n - d$ i.e. that the original sequence $(x_1, \ldots, x_n)$ is d-periodic. Since d divides m_2 and $m_2 < n$ we may conclude that the extended sequence $(a_1, \ldots, a_{n+L})$ is also d periodic. $\square$

Remark: The hypothesis that $m_1 + m_2 \leq n$ is necessary. If we take the sequence of length 10 with period 5

$$(0, 1, 0, 1, 0, 0, 1, 0, 1, 0)$$

and extend this by $(1, 0, 0, 1)$ we obtain the sequence

$$(0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1)$$

of length 14 that has period 7. Obviously the original sequence is not constant even though 5 and 7 are relatively prime.

THEOREM 1. *Let (Σ_A, σ) be a mixing subshift of finite type. Let L be such that $A^L > 2$. Let $n \geq L$. Let $(a_1, \dots, a_n)$ be a periodic sequence. Either*
1. *$(a_1, \dots, a_n)$ has minimal period n or $\frac{n}{2}$, or*
2. *there exists an extension $(a_1, \dots, a_{n+L})$ of $(a_1, \dots, a_n)$ such that $(a_1, \dots, a_{n+L})$ is periodic with minimal period $n + L$ or $\frac{n+L}{2}$.*

Proof: If $(a_1, \dots, a_n)$ has minimal period, m_1, either n or $\frac{n}{2}$ then we are done. Therefore we may suppose that $m_1 \leq \frac{n}{3}$. Let $(a_1, \dots, a_{n+L})$ be an arbitrary periodic extension of $(a_1, \dots, a_n)$. If $(a_1, \dots, a_{n+L})$ has minimal period, m_2, either $n + L$ or $\frac{n+L}{2}$ then we are done. Therefore we may suppose that $m_2 \leq \frac{n+L}{3}$.

In this case, since $L \leq n$ we must have $m_1 + m_2 \leq n$. Hence by our Lemma 1 we are able to show that the extended sequence $(a_1, \dots, a_{n+L})$ has period m_1. There is a unique extension of $(a_1, \dots, a_n)$ that makes $(a_1, \dots, a_{n+L})$ have period m_1 but there are at least two ways of completing $(a_1, \dots, a_{n+L})$. Using this other completion we get $(a_1, \dots, a_{n+L})$ does not have period m_1. If we denote its minimal period by m_2 we see that we must have $m_1 + m_2 > n$. This means that m_2 must be at least $\frac{n+L}{2}$.

Now our main theorem:

THEOREM 2. *Let f be a topologically transitive C^1 Anosov diffeomorphism of a compact connected manifold M. Given $\epsilon > 0$ there exists $D \geq 1$ and $N > 0$ such that for all $x \in M$ and $n \in \mathbb{N}$ there exists a periodic point $p \in M$ with minimal period $m \in \mathbb{N}$ and $d \in \mathbb{N}$ such that:*
1. *for all $0 \leq i \leq n - 1$*
$$d_M(f^i x, f^i p) < \epsilon,$$
and
2. *$n \leq d \cdot m \leq n + N$ and $1 \leq d \leq D$.*

Proof: Let $\epsilon > 0$ be arbitrary. There exists a Markov partition $\mathfrak{M}$ of M by "rectangles" of diameter less than ϵ [**Bow70b**] . Let (Σ_A, σ) be the associated subshift of finite type with transition matrix A and alphabet $\mathscr{A}$. Every transitive Anosov diffeomorphism of a connected manifold is topologically mixing so there exists $L \in \mathbb{N}$ such that A^L is a positive matrix. Immediately we have $A^{2L} \geq 2$. By [**Bow70a**, Proposition 10] there exists $k \in \mathbb{N}$ the canonical projection $\pi : \Sigma_A \to M$ satisfies $\#\pi^{-1}(x) \leq k$ for all $x \in M$.

Consider one of the possible lifts of the point $x \in M$, $(\dots, x_0, x_1, \dots, x_{n-1}, \dots)$. Consider the finite sequence $(x_0, \dots, x_{n-1})$. We can extend this by $2L$ states to get a new finite sequence $(y_0, \dots, y_{n-1+2L})$ that is periodic. We chose to extend by $2L$ rather than simply L so that $2L < n + 2L$. Now we can apply our symbolic extension lemma to obtain either a point q of period $n+2L$ with minimal period at least $\frac{n+2L}{2}$ or a point q of period $n+4L$ with minimal period $\frac{n+4L}{2}$. Let m be in the minimal period of the point q. The orbit of the periodic point q consists of m distinct points. Projecting the orbit under π gives at least m/k distinct points. Hence the minimal period of the projected point $p=\pi(q)$ is at least m/k.

Taking $N=4L$ and $D=2k$ we see that we obtain a periodic point p of period $n \leq n' \leq n+N$ with minimal period at least $\frac{n'}{D}$.

Since we extended the original $(x_0,\ldots,x_{n-1})$ we have that p and x belong to the same rectangle for the first n iterations of f. This means that

$$d_M(f^i x, f^i p) < \epsilon$$

for $0 \leq i \leq n-1$. □

3. *Completing the Proof of Theorem 6.3*

Theorem 6.3 states that if the distortion of f along every periodic orbit is bounded then the distortion of any iterate of f is uniformly bounded. The idea was that any orbit segment is close to a segment of a periodic orbit whose period is not very different from the length of the orbit segment. This lead to the (54) in [**dlLW09**]

$$\begin{aligned} K_{g,E^s}(f^n,p) &\leq K_{g,E^s}(f^{n+\ell},p)\, K_{g,E^s}(f^{-\ell},p) \\ &\leq C_{\mathrm{per}}\, K_{g,E^s}(f^{-\ell}). \end{aligned} \tag{54}$$

Here we used $K_{g,E^s}(f^{n+\ell},p) \leq C_{per}$ since we were supposing that $n+\ell$ was the minimal period of the periodic point p. We are unable to show that this is the case, however using our previous lemma, we may find a periodic point p with minimal period m such that for some $d \in \mathbb{N}$ with $1 \leq d \leq D$ we have $m \cdot d = n+\ell$ for $0 \leq \ell \leq N$. Now we have

$$\begin{aligned} K_{g,E^s}(f^{n+\ell},p) &= K_{g,E^s}(f^{m\cdot d},p) \\ &= K_{g,E^s}(f^m,p)^d \\ &\leq K_{g,E^s}(f^m,p)^D \\ &\leq C_{per}^D \end{aligned}$$

This then leads immediately to our replacement for (54)

$$\begin{aligned} K_{g,E^s}(f^n,p) &\leq K_{g,E^s}(f^{n+\ell},p)\, K_{g,E^s}(f^{-\ell},p) \\ &\leq C_{\mathrm{per}}^D\, K_{g,E^s}(f^{-\ell}). \end{aligned} \tag{54'}$$

With estimate (54') the remainder of the proof of Theorem 6.3 in [**dlLW09**] carries through as stated. The only change is in the value of the constant obtained.

References

[**Bow70a**] Rufus Bowen. Markov partitions and minimal sets for Axiom A diffeomorphisms. *Amer. J. Math.*, 92:907–918, 1970.

[**Bow70b**] Rufus Bowen. Markov partitions for Axiom A diffeomorphisms. *Amer. J. Math.*, 92:725–747, 1970.

[**dlLW09**] Rafael de la Llave and Alistair Windsor. Livšic theorems for non-commutative groups including diffeomorphism groups and results on the existence of conformal structures for Anosov systems. *Ergodic Theory Dynam. Systems*, 2009. Appeared online 17 July 2009 doi: 10.1017/S014338570900039X.

[**FW65**] N. J. Fine and H. S. Wilf. Uniqueness theorems for periodic functions. *Proc. Amer. Math. Soc.*, 16:109–114, 1965.