Avoiding Early Closing:  
A Corrigendum to “Livšic Theorems...”†

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1. Introduction
This paper serves as a corrigendum to our paper [dlLW09]. In particular, in the proof of Theorem 6.3 we claimed the following:

CLAIM: Let \( f \) be a topologically transitive Anosov diffeomorphism of a compact manifold \( M \). For all \( \epsilon > 0 \) there exists \( L \geq 1 \) such that for every \( n \in \mathbb{N} \) and \( x \in M \) there exists a periodic point \( p \in M \) satisfying
1. for all \( 0 \leq i \leq n \),
\[
    d_M(f^i x, f^i p) < \epsilon,
\]
and
2. \( p \) has minimal period \( n + \ell \) with \( 0 \leq \ell \leq L \).

Unfortunately, as B. Kalinin and V. Sadovskaya discovered, the proof sketched contained gaps. Using specification as was suggested in our paper leads to a weaker result than we claimed. In this paper we prove a uniform version of closing:

THEOREM: Let \( f \) be a topologically transitive \( C^1 \) Anosov diffeomorphism of a compact connected manifold \( M \). Given \( \epsilon > 0 \) there exists \( D \geq 1 \) and \( N > 0 \) such that for all \( x \in M \) and \( n \in \mathbb{N} \) there exists a periodic point \( p \in M \) with minimal period \( m \in \mathbb{N} \) and \( d \in \mathbb{N} \) such that:
1. for all \( 0 \leq i \leq n - 1 \)
\[
    d_M(f^i x, f^i p) < \epsilon,
\]
and

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2. \( n \leq d \cdot m \leq n + N \) and \( 1 \leq d \leq D \).

This result is strong enough to complete the proof of Theorem 6.3.

2. Results

To prove our result for Anosov diffeomorphisms we will first prove a similar statement for subshifts of finite type. Since every Anosov diffeomorphism is a factor of a subshift this will allow us to prove the desired result.

Recall that a subshift of finite type can be described by a transition matrix \( A \). Symbol \( j \) may follow symbol \( i \) in a word in \( \Sigma_A \) if \( A_{i,j} = 1 \). A finite sequence \((a_1, \ldots, a_n)\) is called admissible if \( A_{a_i, a_{i+1}} = 1 \) for \( 0 \leq i \leq n - 1 \). We will call a finite sequence \((a_1, \ldots, a_n)\) periodic if it is admissible and \( A_{a_n, a_1} = 1 \) so that the sequence can be extended periodically to a point \( a \in \Sigma_A \) of period \( n \).

The following result is similar to that of Fine and Wilf [FW65].

**LEMMA 1.** Let \((\Sigma_A, \sigma)\) be a subshift of finite type. Let \((a_1, \ldots, a_n)\) be a periodic sequence of period \( m_1 \). Let \((a_1, \ldots, a_n, \ldots, a_{n+L})\) be an extension of \((a_1, \ldots, a_n)\) and be periodic with period \( m_2 \).

If \( m_1 + m_2 \leq n \) then \((a_1, \ldots, a_n)\) and \((a_1, \ldots, a_{n+L})\) are both periodic of period \( \gcd(m_1, m_2) \).

**Proof:** Write \( d := \gcd(m_1, m_2) = k_1 m_1 + k_2 m_2 \) with \( k_1, k_2 \in \mathbb{Z} \). Consider the following variation on the proof of Bézout’s theorem that only uses numbers in the range \( 1, \ldots, n \). If \( k_1 > 0 \) then define \( k_+ = k_1, m_+ = m_1, k_− = −k_2 \) and \( m_− = m_2 \). If \( k_2 > 0 \) then define \( k_+ = k_2, m_+ = m_2, k_− = −k_1 \) and \( m_− = m_1 \).

Let \( 1 \leq i \leq n − d \) be arbitrary and initialize \( k = i \).

1. Add \( m_+ \) to \( k \) successively until either
   (a) adding a further \( m_+ \) would give \( k \) above \( n \), or
   (b) all \( k_+ \) of the \( m_+ \)'s have been used.
2. Subtract \( m_− \) from the new \( k \) successively until either
   (a) subtracting a further \( m_− \) would give \( k \) below \( 1 \), or
   (b) all \( k_− \) of the \( m_− \)'s have been used.
3. If \( k ≠ i + d \) then return to step 1.

Notice that if \( k + m_+ ≥ n + 1 \) and \( k − m_− ≤ 0 \) then \( m_1 + m_2 ≥ n + 1 \) which is a contradiction. Thus this cannot terminate at an intermediate stage and the algorithm must proceed to give \( k = i + d \).

Since each of these steps involves one of the two periods and all of the numbers are within \( 1, \ldots, n \) this shows that \( a_i = a_{i+d} \) for \( 1 ≤ i ≤ n − d \) i.e. that the original sequence \((x_1, \ldots, x_n)\) is \( d \)-periodic. Since \( d \) divides \( m_2 \) and \( m_2 < n \) we may conclude that the extended sequence \((a_1, \ldots, a_{n+L})\) is also \( d \)-periodic. \( \Box \)

**Remark:** The hypothesis that \( m_1 + m_2 ≤ n \) is necessary. If we take the sequence of length 10 with period 5

\[(0, 1, 0, 1, 0, 0, 1, 0, 1, 0)\]
and extend this by \((1,0,0,1)\) we obtain the sequence
\[(0,1,0,1,0,0,1,0,1,0,0,1,0,0,1)\]
of length 14 that has period 7. Obviously the original sequence is not constant even though 5 and 7 are relatively prime.

**Theorem 1.** Let \((\Sigma_A,\sigma)\) be a mixing subshift of finite type. Let \(L\) be such that \(A^L > 2\). Let \(n \geq L\). Let \((a_1, \ldots, a_n)\) be a periodic sequence. Either
1. \((a_1, \ldots, a_n)\) has minimal period \(n\) or \(\frac{n}{2}\), or
2. there exists an extension \((a_1, \ldots, a_{n+L})\) of \((a_1, \ldots, a_n)\) such that \((a_1, \ldots, a_{n+L})\) is periodic with minimal period \(n + L\) or \(\frac{n+L}{2}\).

**Proof:** If \((a_1, \ldots, a_n)\) has minimal period, \(m_1\), either \(n\) or \(\frac{n}{2}\) then we are done. Therefore we may suppose that \(m_1 \leq \frac{n}{2}\). Let \((a_1, \ldots, a_{n+L})\) be an arbitrary periodic extension of \((a_1, \ldots, a_n)\). If \((a_1, \ldots, a_{n+L})\) has minimal period, \(m_2\), either \(n + L\) or \(\frac{n+L}{2}\) then we are done. Therefore we may suppose that \(m_2 \leq \frac{n+L}{2}\).

In this case, since \(L \leq n\) we must have \(m_1 + m_2 \leq n\). Hence by our Lemma 1 we are able to show that the extended sequence \((a_1, \ldots, a_{n+L})\) has period \(m_1\). There is a unique extension of \((a_1, \ldots, a_n)\) that makes \((a_1, \ldots, a_{n+L})\) have period \(m_1\) but there are at least two ways of completing \((a_1, \ldots, a_{n+L})\). Using this other completion we get \((a_1, \ldots, a_{n+L})\) does not have period \(m_1\). If we denote its minimal period by \(m_2\) we see that we must have \(m_1 + m_2 > n\). This means that \(m_2\) must be at least \(\frac{n+L}{2}\).

Now our main theorem:

**Theorem 2.** Let \(f\) be a topologically transitive C\(^1\) Anosov diffeomorphism of a compact connected manifold \(M\). Given \(\epsilon > 0\) there exists \(D \geq 1\) and \(N > 0\) such that for all \(x \in M\) and \(n \in \mathbb{N}\) there exists a periodic point \(p \in M\) with minimal period \(m \in \mathbb{N}\) and \(d \in \mathbb{N}\) such that:
1. for all \(0 \leq i \leq n - 1\)
   \[d_M(f^i x, f^i p) < \epsilon,\]
   and
2. \(n \leq d \cdot m \leq n + N\) and \(1 \leq d \leq D\).

**Proof:** Let \(\epsilon > 0\) be arbitrary. There exists a Markov partition \(\mathcal{M}\) of \(M\) by “rectangles” of diameter less than \(\epsilon\) [Bow70b]. Let \((\Sigma_A,\sigma)\) be the associated subshift of finite type with transition matrix \(A\) and alphabet \(\mathcal{A}\). Every transitive Anosov diffeomorphism of a connected manifold is topologically mixing so there exists \(L \in \mathbb{N}\) such that \(A^L\) is a positive matrix. Immediately we have \(A^{2L} \geq 2\). By [Bow70a, Proposition 10] there exists \(k \in \mathbb{N}\) the canonical projection \(\pi: \Sigma_A \rightarrow M\) satisfies \(\#\pi^{-1}(x) \leq k\) for all \(x \in M\).

Consider one of the possible lifts of the point \(x \in M\), \((\ldots, x_0, x_1, \ldots, x_{n-1}, \ldots)\). Consider the finite sequence \((x_0, \ldots, x_{n-1})\). We can extend this by \(2L\) states to get a new finite sequence \((y_0, \ldots, y_{n-1+2L})\) that is periodic. We chose to extend by \(2L\) rather than simply \(L\) so that \(2L < n + 2L\). Now we can apply our symbolic
extension lemma to obtain either a point $q$ of period $n + 2L$ with minimal period at least $\frac{n + 2L}{2}$ or a point $q$ of period $n + 4L$ with minimal period $\frac{n + 4L}{2}$. Let $m$ be in the minimal period of the point $q$. The orbit of the periodic point $q$ consists of $m$ distinct points. Projecting the orbit under $\pi$ gives at least $m/k$ distinct points. Hence the minimal period of the projected point $p = \pi(q)$ is at least $m/k$.

Taking $N = 4L$ and $D = 2k$ we see that we obtain a periodic point $p$ of period $n \leq n' \leq n + N$ with minimal period at least $\frac{n'}{D}$.

Since we extended the original $(x_0, \ldots, x_{n-1})$ we have that $p$ and $x$ belong to the same rectangle for the first $n$ iterations of $f$. This means that

$$d_M(f^i x, f^i p) < \epsilon$$

for $0 \leq i \leq n - 1$. \hfill \Box

3. Completing the Proof of Theorem 6.3

Theorem 6.3 states that if the distortion of $f$ along every periodic orbit is bounded then the distortion of any iterate of $f$ is uniformly bounded. The idea was that any orbit segment is close to a segment of a periodic orbit whose period is not very different from the length of the orbit segment. This lead to the (54) in [dlILW09]

$$K_{g,E^s}(f^n, p) \leq K_{g,E^s}(f^{n+\ell}, p) K_{g,E^s}(f^{-\ell}, p) \leq C_{\text{per}} K_{g,E^s}(f^{-\ell}).$$

Here we used $K_{g,E^s}(f^{n+\ell}, p) \leq C_{\text{per}}$ since we were supposing that $n + \ell$ was the minimal period of the periodic point $p$. We are unable to show that this is the case, however using our previous lemma, we may find a periodic point $p$ with minimal period $m$ such that for some $d \in \mathbb{N}$ with $1 \leq d \leq D$ we have $m \cdot d = n + \ell$ for $0 \leq \ell \leq N$. Now we have

$$K_{g,E^s}(f^{n+\ell}, p) = K_{g,E^s}(f^{m \cdot d}, p) = K_{g,E^s}(f^m, p)^d \leq K_{g,E^s}(f^m, p)^D \leq C_{\text{per}}^D.$$

This then leads immediately to our replacement for (54)

$$K_{g,E^s}(f^n, p) \leq K_{g,E^s}(f^{n+\ell}, p) K_{g,E^s}(f^{-\ell}, p) \leq C_{\text{per}}^D K_{g,E^s}(f^{-\ell}).$$

With estimate (54’) the remainder of the proof of Theorem 6.3 in [dlILW09] carries through as stated. The only change is in the value of the constant obtained.

References

