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AN APPLICATION OF TOPOLOGICAL MULTIPLE RECURRENCE TO TILING

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Abstract. We show that given any tiling of Euclidean space, any geometric patterns of points, we can find a patch of tiles (of arbitrarily large size) so that copies of this patch appear in the tiling nearly centered on a scaled and translated version of the pattern. The rather simple proof uses Furstenberg's topological multiple recurrence theorem.

1. **Introduction.** Tilings have a very long story as an applied sub-discipline at the service of architecture and decoration [7].

The modern mathematical theory of tiling (we collect several of the standard definitions of tilings in Section 2) can be said to begin in the 60's with the work [17]. The surprising main result of [17] is that there is no algorithmic way to prove that certain collections of tiles in $\mathbb{R}^d$ could be used to tile the whole of $\mathbb{R}^d$, see also [1]. Central to the proof is the construction of sets of tiles that tile the plane only *non-periodically*, such sets of tiles are called *aperiodic*. By now many aperiodic sets of tiles are known of varying complexity [7]. Perhaps the most famous aperiodic set of tiles are the Penrose tiles [8].

The goal of this paper is sort of complementary with the previous results. Rather than showing that there are aperiodic sets of tiles, or sets of tiles that exhibit unordered behavior [3], we show that for any tiling, there has to be some amount of order, in particular approximate periodicity. Roughly speaking, our results show (see Theorems 2, 3, 4 for precise formulations) that in *any* tiling, given *any* pattern and *any* size R, we can find a patch of size at least R so that copies of this patch appear in the tiling nearly centered on a scaled and translated copy of the specified pattern. We can also require that the scalings of the pattern appearing in the conclusions have scaling factors belonging to some special sets (see Section 3.1). For an informal pictorial explanation of our results refer to Section 4.

There are several classes of tilings that have been studied in the literature (see Section 2). Roughly, a tiling is an arrangement of tiles (all of which are copies of some prototiles) that covers the Euclidean space without overlapping. The classes of tilings differ in whether when generating the tiles out of the prototiles one allows rotations, and in whether or not the tiling has a property "finite local complexity" that roughly indicates that the tiles cannot slide.

Even if the results are slightly different in all these cases, the ideas of the proofs are very similar. In all cases, we define a tiling space by considering the closure of all the translations of the given tiling in the appropriate topology (this topology depends on the considerations alluded to in the previous paragraph). One then shows that translations are commuting homeomorphisms of this tiling space and one can use Birkhoff's multiple recurrence theorem. The statements of the results are obtained by unraveling the previous definitions of the tiling spaces and the meaning of convergence in these spaces.

Our proof mirrors Furstenberg's proof of Gallai's theorem using the Birkhoff multiple recurrence theorem [4]. Gallai's theorem is a higher dimensional version of Van der Waerden's theorem [9]. We recall that Van der Waerden's theorem states that any partition of the integers into a finite number of sets has arbitrarily long arithmetic progressions contained in one of the sets of the partition.

The analogy of tilings with dynamical systems in $\mathbb{Z}^d$ was noted in [15]. The application of ergodic theory to tilings was surveyed in [11] and [12]. The use of recurrence (either topological or measure-theoretic) to obtain combinatorial or Ramsey theory results is surveyed in [4].

Let us emphasize that the results of this paper show that all patterns can be found in any tiling of the Euclidean space. Sometimes the fact that arbitrary sets (or graphs) cannot avoid having some amount of order is called *Ramsey theory* [6].

2. Tiling Spaces and the Tiling Topology.

We reproduce here the requisite definitions of tiling spaces and their topologies. We refer the reader to the excellent survey [13] for further exposition and examples. We have augmented the definitions to allow for pinwheel and more general tilings following [10].

2.1. Tiling Spaces.

A set $D \subset \mathbb{R}^d$ is called a *tile* if it is compact and equal to the closure of its interior. A *tiling* of $\mathbb{R}^d$ is a collection of tiles $\{D_i\}$ such that

1. $\bigcup_i D_i = \mathbb{R}^d$ – we say that the tiling *covers* $\mathbb{R}^d$.
2. for all i, j with $i \neq j$, $D_i^\circ \cap D_j^\circ = \varnothing$ – we say that the tiling *packs* $\mathbb{R}^d$.

Two tiles are called *equivalent* if one is a translation of the other. An equivalence class is called a *prototile*. Two tiles are called *congruent* of they are related by an orientation preserving isometry. We will call the equivalence class of congruent tiles a *congruence prototile*.

Definition 1. *Let $\mathcal{T}$ be a finite set of distinct prototiles (resp. congruence prototiles) in $\mathbb{R}^d$. We call the tiling space $X_{\mathcal{T}}$ associated to $\mathcal{T}$ the set of all tilings of $\mathbb{R}^d$ by tiles that are elements of the prototiles (resp. congruence prototiles) in $\mathcal{T}$.*

In both cases the translation of a tiling is still a tiling and hence there is a natural action of $\mathbb{R}^d$ on $X_{\mathcal{T}}$. We shall use $T_{\vec{v}}$ to denote both the translation by $\vec{v} \in \mathbb{R}^d$ and its induced action on $X_{\mathcal{T}}$. If $\mathcal{T}$ consists of congruence prototiles then there is a natural action of the group of orientation preserving isometries on $X_{\mathcal{T}}$.

Definition 2. *Let $\mathcal{T}$ be a finite set of distinct (congruence) prototiles in $\mathbb{R}^d$. A* $\mathcal{T}$-patch *is subset x' of a tiling x such that the union of tiles in x' is a connected subset of $\mathbb{R}^d$.*[1] *We call the union of tiles in x' the support of the patch x', denoted* $\operatorname{supp}(x')$. *We call two patches equivalent if one is a translation of the other and call an equivalence class a* protopatch. *We call two patches congruent if they are related by an orientation preserving isometry and call an equivalence class of congruent patches a* congruence protopatch. *If $\mathcal{T}$ consists of prototiles (resp. congruence prototiles) then we denote by $\mathcal{T}^{(n)}$ the collection of protopatches (resp. congruence protopatches) consisting of n-tiles.*

[1] Some authors require that the union be a simply connected subset of $\mathbb{R}^d$, but we will not use that.

Several elaborations on this definition can be found in the literature including distinguishing identical prototiles by coloring or labeling. Our results can be adapted to such finite extensions but we will not consider them here.

2.2. **Complexity of tilings.**

We emphasize that according to our definition tilings have either a finite number of prototiles or a finite number of congruence prototiles.

In the following subsections, we will consider three situations of increasing generality.

2.2.1. *Finite Local Complexity under Translation.*

Definition 3. *We say that a tiling space has* finite local complexity under translation *if $\mathcal{T}$ consists of a finite number of prototiles and $\mathcal{T}^{(2)}$ is finite. That is, the set of pairs of adjacent tiles is finite.*

One example of a tiling that does not have finite local complexity is the tiling of the plane by a single translated square. The tiles are necessarily located so that there are countably many straight line that consists of entirely of tile edges, these are called shear lines. On each side of a shear line the tiling can be translated along the shear line and these translations can be chosen independently. Hence for any square along the line there are uncountably many configuration of nearby tiles.

Often a finite $\mathcal{T}^{(2)}$ is specified in the description of the tiling space. This describes a set of local matching rules. These local matching rules are often specified in terms of edge labels or colors. These local matching rules can be disposed of by suitably modifying the prototiles, adding little teeth that make them to match only on very precise locations. Wang tiles, the classical Penrose kite and dart tiling, and the tiling by Penrose rhombs all have such local matching rules. In all these cases the prototiles are polygonal. There are local matching rules that require the polygons to meet full edge to edge and, in the case of the Penrose tilings, to preclude the formation of shear lines.

2.2.2. *Finite Local Complexity under the Euclidean Group.*

Definition 4. *We say that a tiling space has* finite local complexity under the Euclidean group *if $\mathcal{T}$ consists of a finite number of congruence prototiles and $\mathcal{T}^{(2)}$ is finite.*

The pinwheel tessellation of J. H. Conway was shown to be a tiling of finite local complexity under the full Euclidean group in [10].

2.2.3. *Tilings without Finite Local Complexity.*

Definition 5. *We say that a tiling space does not have* finite local complexity *if $\mathcal{T}$ consists of a finite number of congruence prototiles and $\mathcal{T}^{(2)}$ is infinite.*

The simplest example of this type of tiling is the tiling by squares, discussed above.

2.3. **The Tiling Topologies.**

In this section we will discuss several variations of the topology of tilings (which are induced by a metric). There are several variations in the literature. All of them have in common the fact that two tilings are at small distance when they are very similar in a large ball about the origin.

2.3.1. *The General Metric.* We first give a definition of a metric that applies to all three of our tiling situations. This is the metric used in [11, Page 70] with proofs in [16].

Given two tilings $x, x' \in X_{\mathcal{T}}$ we define $d(x, x')$ by

$$d(x, x') = \sup_{n \in \mathbb{N}} \frac{1}{n} d_H\big(B_n(\partial x), B_n(\partial x')\big) \tag{1}$$

where d_H denotes the usual Hausdorff distance between compact subsets of $\mathbb{R}^d$, $B_n(\partial x) = B_n \cap \partial x$, $B_n := \{p \in \mathbb{R}^d : \|p\| \leq n\}$, and $\partial x = \cup_{D \in x} \partial D$. In other words, two tilings are close if their skeletons are close on a large ball about the origin.

The following result is proved in [10, 16].

Lemma 1. *The metric makes the tiling space complete and compact. Moreover, the action of $\mathbb{R}^d$ by translation on the tiling space is continuous.*

Proving compactness consists of showing that given a sequence, one can extract a convergent subsequence. Given one sequence of tilings, if we consider any ball, we can extract a subsequence so that the tiles in the ball converge. Note that the positions of the tiles in the ball are given by a finite number of real parameters, which lie on a bounded set. Then, going to a larger ball, we can extract another sequence that converges in both balls. We can repeat the argument over an increasing sequence of balls and then perform a diagonal argument. It is easy to verify that the resulting object is indeed a tiling (if it was not, some violation of the conditions would happen in a finite ball). It is also easy to verify that the diagonal sequence indeed converges in the metric.

The verification of continuity of translations is straightforward. We just note than, if two tilings are very similar in a large ball about the origin, then the two tilings that result from applying a small translation will also be very similar in a large ball about the origin.

Note that the proof indicated here does not use local complexity. It only uses the fact that the position of all the tiles in a ball is indicated by a finite number of parameters.

2.4. **Adapted Metrics.** Though the previous metric applies equally well to all three situations we will give separate metrics for each of the three situations. Each of these metrics is equivalent to the general metric but gives more geometric information.

2.4.1. *Finite Local Complexity under Translation.* We will define a metric d_1 on a tiling space that has finite local complexity under the action of translation that makes two tilings close if after a small translation they agree on a large ball about the origin [14].

Let $K \subset \mathbb{R}^d$ be compact. We denote by $x[[K]]$ the collection of patches x' contained in x with the property that $\operatorname{supp}(x') \supseteq K$. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^d$ and $B_r = \{p \in \mathbb{R}^d : \|p\| \leq r\}$.

Definition 6. *We define a metric on the tiling space $X_{\mathcal{T}}$ by*

$$d_1(x, y) = \inf\Big(\{\frac{1}{\sqrt{2}}\} \cup \{0 < r < \frac{1}{\sqrt{2}} : \exists x' \in x[[B_{\frac{1}{r}}]], \\ y' \in y[[B_{\frac{1}{r}}]], \vec{v} \in B_r, \text{ such that } T_{\vec{v}}\, x' = y'\}\Big).$$

The proof that this defines a metric may be found in [13] and does not depend on finite local complexity. The only property that is not immediately obvious is the triangle inequality. The bound of $\frac{1}{\sqrt{2}}$ is used precisely in the verification of the triangle inequality. The rôle that finite local complexity plays is summarized by the following theorem of Rudolph [15].

Lemma 2. *Any tiling space $X_{\mathcal{T}}$ endowed with d_1 is complete. If $X_{\mathcal{T}}$ has finite local complexity under translation then $X_{\mathcal{T}}$ endowed with the metric d_1 is compact. Moreover, the action of $\mathbb{R}^d$ on $X_{\mathcal{T}}$ by translation is continuous.*

2.4.2. *Finite Local Complexity under the Euclidean Group.* We will define a metric d_2 on a tiling space with finite local complexity under the full Euclidean group that makes two tilings close if after a small isometry they agree on a large ball about the origin.

We define a metric on the group of direct isometrics of $\mathbb{R}^d$

$$\mathcal{E}^d := \{T\vec{p} = A\vec{p} + \vec{b} : A \in SO(d), \vec{b} \in \mathbb{R}^d\}$$

by

$$d_{\mathcal{E}^d}(A_1\vec{p} + \vec{b}_1, A_2\vec{p} + \vec{b}_2) := \max\{\|A_1 - A_2\|, \|\vec{b}_1 - \vec{b}_2\|\}.$$

Using a common abuse of notation we can write

$$B_r(\mathcal{E}^d) = \{T \in \mathcal{E}^d : d_{\mathcal{E}^d}(T, \mathrm{Id}) < r\} = \{T\vec{p} = A\vec{p} + \vec{b} : \|A - \mathrm{Id}\| < r, \|\vec{b}\| < r\}.$$

Using these notations we define a metric in a fashion similar to d_1.

Definition 7. *We define a metric on the tiling space $X_{\mathcal{T}}$ by*

$$d_2(x, y) = \inf\big(\{\frac{1}{\sqrt{2}}\} \cup \{0 < r < \frac{1}{\sqrt{2}} : \exists x' \in x[[B_{\frac{1}{r}}]],$$
$$y' \in y[[B_{\frac{1}{r}}]], T \in B_r(\mathcal{E}^d), \text{ such that } T\,x' = y'\}\big).$$

The proof that this is a metric follows almost exactly the proof in [13] for d_1.

Under this definition any tiling space is complete and the action of translations is continuous. If the tiling space has finite local complexity under the Euclidean group then the tiling space endowed with the metric d_2 is compact. The argument is the same as that sketched for d following Lemma 1.

2.4.3. *Tilings without Finite Local Complexity.* If the tiling space does not have finite local complexity then metrics which focus on motions of the whole tiling will not give compactness of the tiling space. In this case we define two tilings to be close if they agree on a large ball about the origin after a small motion of each individual tile.

Definition 8. *We define a metric on the tiling space $X_{\mathcal{T}}$ by*

$$d_3(x, y) = \inf\big(\{\frac{1}{\sqrt{2}}\} \cup \big\{0 < r < \frac{1}{\sqrt{2}} : \exists \{t_i\}_{i=1}^m \in x[[B_{\frac{1}{r}}]],$$
$$\{s_i\}_{i=1}^m \in y[[B_{\frac{1}{r}}]], \{T_i\}_{i=1}^m \subset B_r(\mathcal{E}^d), \text{ such that } T_i\,t_i = s_i \text{ for } i = 1, \ldots, m\ \big\}\big).$$

Under this definition any tiling space is compact and the action of translations is continuous. The proof that this is a metric follows almost exactly the proof in [13] for d_1.

3. Topological Multiple Recurrence and its Application to Tiling Theory.

The following is the Multiple Birkhoff Recurrence Theorem for commuting homeomorphisms [4, Proposition 2.5]. The same result holds for commuting continuous maps and can be obtained from this one by passing to the natural extension, but for our applications, we only require the version for homeomorphisms.

Theorem 1. *Let X be a compact metric space and $T_1, \ldots, T_l$ commuting homeomorphisms of X. Then there exists a point $x \in X$ and a sequence $n_k \to \infty$ such that $T_i^{n_k} x \to x$ simultaneously for $i = 1, \ldots, l$.*

Furstenberg's original application of this theorem was to prove Gallai's extension of the Van der Waerden's theorem to higher dimensions [9].[2]

Our proof follows the same scheme as Furstenberg's proof of Gallai's theorem [4, Theorem 2.7] though the structure of our topological spaces is quite different.

Theorem 2 (Main Theorem for Tilings with Finite Local Complexity under Translation)**.** *Let $X_{\mathcal{T}}$ be an $\mathbb{R}^d$-tiling space with finite local complexity under translation, and let $x \in X_{\mathcal{T}}$ be an arbitrary tiling. Given $\epsilon > 0$ and a finite subset $F \subset \mathbb{R}^d$ there exists an $n \in \mathbb{N}$, and a patch p contained in x such that*

1. $\operatorname{supp}(p)$ *contains a ball of radius $\frac{1}{\epsilon}$ (not necessarily centered at the origin),*
2. *for each $\vec{u} \in F$ there exists a vector $\vec{c}$ with $\|\vec{c}\| < \epsilon$ such that*

$$T_{n\vec{u}+\vec{c}}\, p \subset x.$$

Proof. Consider the set

$$X = \overline{\{T_{\vec{v}}\, x : \vec{v} \in \mathbb{R}^d\}}$$

where the closure is taken according to the topology induced by the metric d_1. X is a translation invariant compact subset of $X_{\mathcal{T}}$. Let $F = \{\vec{u}_1, \ldots, \vec{u}_l\}$ and consider the l commuting homeomorphisms of X given by $T_i = T_{-\vec{u}_i}$.

By Theorem 1 there exists a point $y \in X$ and a sequence $n_k \to \infty$ such that $T_i^{n_k} y \to y$ for $1 \le i \le l$. In particular for large enough n_k we have $d_1(T_i^{n_k} y, y) < \epsilon$. Since $y \in X$ is either a translation of x or the limit of translations of x we can find $\vec{v} \in \mathbb{R}^d$ such that $d_1(T_{-n_k\vec{u}_i - \vec{v}}\, x, T_{-\vec{v}}\, x) < \epsilon$. By definition of the metric there exists $x' \in (T_{-n_k\vec{u}_i-\vec{v}}\, x)[[B_{\frac{1}{\epsilon}}]]$, $p_i \in (T_{-\vec{v}}\, x)[[B_{\frac{1}{\epsilon}}]]$, and a vector $\vec{c}_i \in \mathbb{R}^d$ with $\|\vec{c}_i\| < \epsilon$ such that $T_{-\vec{c}_i} x' = p_i$. Now consider p' to be the connected component of $\cap_{i=1}^{l} p_i$ that contains the origin. Since each p_i has $B_{\frac{1}{\epsilon}} \subset \operatorname{supp} p_i$ we see that $B_{\frac{1}{\epsilon}} \subset \operatorname{supp} p'$. Now we define $p = T_{\vec{v}}\, p'$. By construction $p \subset T_{-n_k\vec{u}_i - \vec{c}_i}\, x$ and thus we get

$$T_{n_k\vec{u}_i + \vec{c}_i}\, p \subset x$$

as required. □

The only things we have used crucially are the compactness of the space and the continuity of the action by translation. For completeness we give proofs for the remaining two cases though we emphasize that the crucial step is the same in all the proofs.

Theorem 3 (Main Theorem for Tilings with Finite Local Complexity under the Euclidean Group)**.** *Let $X_{\mathcal{T}}$ be an $\mathbb{R}^d$-tiling space of finite local complexity, and let $x \in X_{\mathcal{T}}$ be an arbitrary tiling. Given $\epsilon > 0$ and a finite subset $F \subset \mathbb{R}^d$ there exists an $n \in \mathbb{N}$, a point $\vec{v} \in \mathbb{R}^d$, and a patch p contained in x such that*

[2]Gallai is also known as Grünwald.

1. $B_{\frac{1}{\epsilon}}(\vec{v}) \subset \operatorname{supp}(p)$,
2. *for each* $\vec{u} \in F$ *there exists an* $S \in B_\epsilon(\mathcal{E}^d)$ *such that*

$$T_{n\vec{u}+\vec{v}}\, S\, T_{-\vec{v}}\, p \subset x.$$

Proof. Consider the set

$$X = \overline{\{T_{\vec{v}}\, x : \vec{v} \in \mathbb{R}^d\}}$$

where the closure is taken according to the topology induced by the metric d_2. X is a translation invariant compact subset of $X_{\mathcal{T}}$. Let $F = \{\vec{u}_1, \ldots, \vec{u}_l\}$ and consider the l commuting homeomorphisms of X given by $T_i = T_{-\vec{u}_i}$.

By Theorem 1 there exists a point $y \in X$ and a sequence $n_k \to \infty$ such that $T_i^{n_k} y \to y$ for $1 \le i \le l$. In particular for large enough n_k we have $d_2(T_i^{n_k} y, y) < \epsilon$. Since $y \in X$ is either a translation of x or the limit of translations of x we can find $\vec{v} \in \mathbb{R}^d$ such that $d_2(T_{-n_k\vec{u}_i-\vec{v}}\, x, T_{-\vec{v}}\, x) < \epsilon$. By definition of the metric there exists $x' \in (T_{-n_k\vec{u}_i-\vec{v}}\, x)[[B_{\frac{1}{\epsilon}}]]$, $p_i \in (T_{-\vec{v}}\, x)[[B_{\frac{1}{\epsilon}}]]$, and an isometry $S_i \in B_\epsilon(\mathcal{E}^d)$ such that $S_i^{-1} x' = p_i$. Now consider p' to be the connected component of $\cap_{i=1}^{l} p_i$ that contains the origin. Since each p_i has $B_{\frac{1}{\epsilon}} \subset \operatorname{supp} p_i$ we see that $B_{\frac{1}{\epsilon}} \subset \operatorname{supp} p'$. Now we define $p = T_{\vec{v}}\, p'$. By construction

$$S_i\, T_{-\vec{v}}\, p = S_i\, p' \subset S_i\, p_i = x'$$

and thus we get

$$T_{n_k\vec{u_i}+\vec{v}}\, S_i\, T_{-\vec{v}}\, p \subset T_{n_k\vec{u_i}+\vec{v}}\, x' \subset x$$

as required. □

Theorem 4 (Main Theorem for Tilings without Finite Local Complexity)**.** *Let* $X_{\mathcal{T}}$ *be an* $\mathbb{R}^d$*-tiling space with* $\mathcal{T}$ *finite and let* $x \in X_{\mathcal{T}}$ *be an arbitrary tiling. Given* $\epsilon > 0$ *and a finite subset* $F \subset \mathbb{R}^d$ *there exists an* $n \in \mathbb{N}$, $\vec{v} \in \mathbb{R}^d$, *and a patch* $p = \{t_i\}_{i=1}^m$ *contained in* x *such that*

1. $B_{\frac{1}{\epsilon}}(\vec{v}) \subset \operatorname{supp}(p)$,
2. *for each* $\vec{u} \in F$ *and* $1 \le i \le m$ *there exists* $S_i \in B_\epsilon(\mathcal{E}^d)$ *such that*

$$T_{n\vec{u}+\vec{v}}\, S_i\, T_{-\vec{v}}\, t_i \in x.$$

Proof. Consider the set

$$X = \overline{\{T_{\vec{v}}\, x : \vec{v} \in \mathbb{R}^d\}}$$

where the closure is taken according to the topology induced by the metric d_3. X is a translation invariant compact subset of $X_{\mathcal{T}}$. Let $F = \{\vec{u}_1, \ldots, \vec{u}_l\}$ and consider the l commuting homeomorphisms of X given by $T_i = T_{-\vec{u}_i}$.

By Theorem 1 there exists a point $y \in X$ and a sequence $n_k \to \infty$ such that $T_i^{n_k} y \to y$ for $1 \le i \le l$. In particular for large enough n_k we have $d_3(T_i^{n_k} y, y) < \epsilon$. Since $y \in X$ is either a translation of x or the limit of translations of x we can find $\vec{v} \in \mathbb{R}^d$ such that $d_3(T_{-n_k\vec{u}_i-\vec{v}}\, x, T_{-\vec{v}}\, x) < \epsilon$. By definition of the metric there exists $x' = \{s_{i,j}\}_{j=1}^{m_i} \in (T_{-n_k\vec{u}_i-\vec{v}}\, x)[[B_{\frac{1}{\epsilon}}]]$, $p_i = \{t_j\}_{j \in J_i} \in (T_{-\vec{v}}\, x)[[B_{\frac{1}{\epsilon}}]]$, and isometries $\{S_{i,j}\}_{j \in J_i} \in B_\epsilon(\mathcal{E}^d)$ such that $S_i^{-1} s_{i,j} = t_j$. Now consider p' to be the connected component of $\cap_{i=1}^{l} p_i$ that contains the origin. Thus $p' = \{t_j\}_{j \in J}$ for some $J \subset \cap_{i=1}^{l} J_i$. Since each p_i has $B_{\frac{1}{\epsilon}} \subset \operatorname{supp} p_i$ we see that $B_{\frac{1}{\epsilon}} \subset \operatorname{supp} p'$. Now we define $p = T_{\vec{v}}\, p' = \{\tau_j\}_{j \in J} \subset x$. By construction

$$S_{i,j}\, T_{-\vec{v}}\, \tau_j = S_{i,j}\, t_j = s_{i,j}$$

and thus we get

$$T_{n_k\vec{u}_i+\vec{v}}\, S_i\, T_{-\vec{v}}\, \tau_j = T_{n_k\vec{u}_i+\vec{v}}\, s_{i,j} \in x$$

as required. □

3.1. **IP-Sets and Dilation Factors.** There is a refinement of our main theorems that allows some control over the dilation factors that appear. Unfortunately it does not give any information on the size of the dilation required.

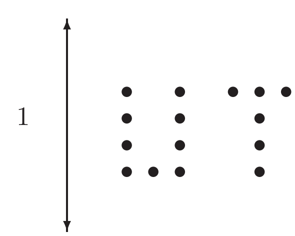
Definition 9. *A set of positive integers R is called an IP-set if there exists a sequence $(p_i)_{i=1}^{\infty}$ of natural numbers such that R consists of the numbers p_i together with all finite sums*

$$p_{i_1} + p_{i_2} + \cdots + p_{i_k}$$

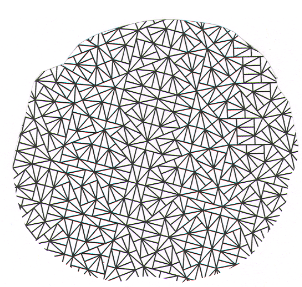
with $i_1 < i_2 < \cdots < i_k$.

IP-sets appear naturally in situations where recurrence plays a central rôle [5] [2]. The dilation factor n in our proofs arises from an application of the Birkhoff Multiple Recurrence Theorem, Theorem 1. It is shown in [4, Theorem 2.18], that one can restrict the sets of numbers that appear in the conclusion of the Birkhoff Multiple Recurrence Theorem 1 to an a priori given IP-set. Hence Theorem 2, Theorem 3, and Theorem 4 hold true when we restrict the number n to lie in some a priori specified IP-set without any modification of the proofs.

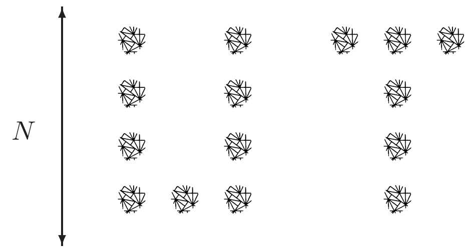
4. **An informal pictorial illustrations of the results.** For example, given a tiling with finite local complexity (of either type), a finite set of points in $\mathbb{R}^n$, e.g.,



a size R, and a $\varepsilon > 0$. We can find a patch containing a ball of radius R



and a number N, such that the configuration



appears somewhere in the tiling up to an isometry of size less than ε. The appearance of this configuration may be very far from the origin.

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REFERENCES

[1] Robert Berger. The undecidability of the domino problem. *Mem. Amer. Math. Soc. No.*, 66:72, 1966.
[2] Vitaly Bergelson, Hillel Furstenberg, and Randall McCutcheon. IP-sets and polynomial recurrence. *Ergodic Theory Dynam. Systems*, 16(5):963–974, 1996.
[3] John H. Conway and Charles Radin. Quaquaversal tilings and rotations. *Invent. Math.*, 132(1):179–188, 1998.
[4] H. Furstenberg. *Recurrence in ergodic theory and combinatorial number theory*. Princeton University Press, Princeton, N.J., 1981. M. B. Porter Lectures.
[5] H. Furstenberg and B. Weiss. Simultaneous Diophantine approximation and IP-sets. *Acta Arith.*, 49(4):413–426, 1988.
[6] Ronald L. Graham, Bruce L. Rothschild, and Joel H. Spencer. *Ramsey theory*. John Wiley & Sons Inc., New York, 1980. Wiley-Interscience Series in Discrete Mathematics, A Wiley-Interscience Publication.
[7] Branko Grünbaum and G. C. Shephard. *Tilings and patterns*. W. H. Freeman and Company, New York, 1987.
[8] Roger Penrose. Pentaplexy. *Bull. Inst. Math. Appl.*, ‘0:266–271, 1974.
[9] R. Rado. Verallgemeinerung eines satzes von van der waerden mit anwendungen auf ein problem der zahlentheorie. *Sonderausg. Sitzungsber. Preuss. Akad. Wiss. Phys.–Math. Klasse*, 17:1–10, 1933.
[10] Charles Radin. The pinwheel tilings of the plane. *Ann. of Math. (2)*, 139(3):661–702, 1994.
[11] Charles Radin. Miles of tiles. In *Ergodic theory of* $\mathbf{Z}^d$ *actions (Warwick, 1993–1994)*, volume 228 of *London Math. Soc. Lecture Note Ser.*, pages 237–258. Cambridge Univ. Press, Cambridge, 1996.
[12] Charles Radin. *Miles of tiles*, volume 1 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 1999.
[13] E. Arthur Robinson, Jr. Symbolic dynamics and tilings of $\mathbb{R}^d$. In *Symbolic dynamics and its applications*, volume 60 of *Proc. Sympos. Appl. Math.*, pages 81–119. Amer. Math. Soc., Providence, RI, 2004.
[14] Daniel J. Rudolph. Rectangular tilings of $\mathbf{R}^n$ and free $\mathbf{R}^n$-actions. In *Dynamical systems (College Park, MD, 1986–87)*, volume 1342 of *Lecture Notes in Math.*, pages 653–688. Springer, Berlin, 1988.
[15] Daniel J. Rudolph. Markov tilings of $\mathbf{R}^n$ and representations of $\mathbf{R}^n$ actions. In *Measure and measurable dynamics (Rochester, NY, 1987)*, volume 94 of *Contemp. Math.*, pages 271–290. Amer. Math. Soc., Providence, RI, 1989.

[16] Charles Radin and Mayhew Wolff. Space tilings and local isomorphism. *Geom. Dedicata*, 42(3):355–360, 1992.

[17] Hao Wang. Proving theorems by pattern recognition-II. *Bell System Tech. Journal*, 40(1):1–41, 1961.

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