Ergod. Th. & Dynam. Sys. (2007), **1**, 1
Printed in the United Kingdom

Nonstandard Smooth Realizations of Liouville Rotations

B. R. Fayad†, M. Saprykina‡ and A. Windsor§ †

† *LAGA, UMR 7539, Université Paris 13, 93430 Villetaneuse, France*
‡*Department of Math & Stats, Jeffery Hall, University Ave. Kingston, ON Canada, K7L 3N6*
§*Department of Mathematics, University of Texas at Austin, 1 University Station, C1200, Austin, TX 78712-0257, USA*

(*Received*)

Abstract. We augment the method of C^∞ conjugation approximation with explicit estimates on the conjugacy map. This allows us to construct ergodic volume preserving diffeomorphisms measure-theoretically isomorphic to any a priori given Liouville rotation on a variety of manifolds. In the special case of tori the maps can be made uniquely ergodic.

1. *Introduction*

We call a diffeomorphism f of a compact manifold M that preserves a smooth measure μ a *smooth realization* of an abstract system (X, T, ν) if they are measure-theoretically isomorphic. A diffeomorphism of a compact manifold has finite entropy with respect to any Borel measure. The natural question therefore becomes whether every finite entropy automorphism of a Lebesgue space has a smooth realization. This problem remain stubbornly intractable and there remain abstract examples that have no known smooth realizations.

We seek to find smooth realizations of one of the simplest types of automorphisms; aperiodic automorphisms with pure point spectrum with a group of eigenvalues with a single generator. Such automorphisms are measure theoretically isomorphic to irrational rotations of the circle. They therefore have a natural

† The authors would like to thank Anatole Katok and Jean-Paul Thouvenot for their interest in this paper and their invaluable support and encouragement during its extended nascency.
M. Saprykina would like to thank the University of Texas at Austin and Rafael de la Llave for support and hospitality.
A. Windsor gratefully acknowledges the generous support of the Université Paris 13 and the CNRS as well as the hospitality of the Laboratoire de Probabilités et Modèles Aléatoires.

smooth realization. We seek smooth realizations on manifolds other than $\mathbb{T}$. Such realizations are called non-standard smooth realizations.

We extend the conjugation approximation method of Anosov and Katok [**1**] to construct non-standard smooth realizations of a given Liouville rotation on $\mathbb{T}$ on a variety of manifolds M. Indeed, in the special case that the manifold is $\mathbb{T}^d$ for $d \geq 2$, we can produce uniquely ergodic realizations of the given Liouville rotation. The crucial new ingredient is an explicit construction of the conjugating maps that allows us to estimate their derivatives. This allows us to ensure that the construction converges for a predetermined Liouville number α. The approach parallels that taken in [**3**]. The original paper of Anosov and Katok paper constructed non-standard smooth realizations of a dense set of Liouville rotations. However, without estimates, it was not possible to identify which Liouville rotations could be realized.

Definition 1. *A number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is a Liouville number if for all $k > 0$ we have*

$$\liminf_{q\to\infty} q^k \|q\alpha\| = 0 \tag{1}$$

where $\|q\alpha\| = \inf_{p\in\mathbb{Z}} |q\alpha - p|$.

Let $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ denote the d-dimensional torus. Let $R_\theta : \mathbb{T} \to \mathbb{T}$ be the rotation of the circle, taken with the Haar probability measure, given by $R_\theta(x) = x + \theta \mod 1$.

Denote by $\mathrm{Diff}^\infty(M, \mu)$ the class of C^∞ diffeomorphisms of M that preserve a C^∞ smooth volume μ. Throughout this paper we will use λ for the probability measure induced by the standard Lebesgue measure.

Theorem 1. *Let M be a compact connected manifold of dimension $d \geq 2$, possibly with boundary, that admits an effective C^∞ action of $\mathbb{T}$ preserving a C^∞ smooth volume μ. For every $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ Liouville there exists an ergodic $T \in \mathrm{Diff}^\infty(M, \mu)$ measure-theoretically isomorphic to the rotation R_α.*

In the special case $M = \mathbb{T}^d$ we can strengthen the result to obtain unique ergodicity.

Theorem 2. *For every Liouville $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, and every $d \geq 2$ there exists a uniquely ergodic transformation $T \in \mathrm{Diff}^\infty(\mathbb{T}^d, \lambda)$ such that T is measure-theoretically isomorphic to the rotation R_α.*

It remains open whether there are C^∞ realizations of Diophantine rotations on any manifold other than $\mathbb{T}$.

2. *Construction*

2.1. *Outline* The required measure preserving diffeomorphism T is constructed as the limit of a sequence of periodic measure preserving diffeomorphisms T_n. For each of the properties that we wish the limiting diffeomorphism T to possess, we establish an appropriate finitary version possessed by the periodic diffeomorphism T_n.

Let $S : \mathbb{T} \times M \to M$ denote an effective C^∞ action of $\mathbb{T}$ on M that preserves the volume and denote by S_α the diffeomorphism $S(\alpha, \cdot)$. The diffeomorphism T_n is given by

$$T_n := H_n S_{\alpha_n} H_n^{-1} \tag{2}$$

where $\alpha_n \in \mathbb{Q}$ and $H_n \in \mathrm{Diff}^\infty(M, \lambda)$.

We choose a sequence $\alpha_n := p'_n/q'_n$ such that $|\alpha_n - \alpha| \to 0$ monotonically. This choice defines a sequence of intermediate scales by $q_n = q_{n-1}^d q'_n$ satisfying $q'_n < q_n < q'_{n+1}$ which are geometrically natural for all the previous transformations. Fixing q_n determines H_{n+1} via the iterative formula

$$H_{n+1} = H_n h_{n,q_n}. \tag{3}$$

Defining the family of maps $h_{n,q}$ and investigating their properties will form the bulk of this paper.

2.2. *Reduction* Though Theorem 1 appears considerably more general than Theorem 2 they follow from nearly identical arguments. We are able to reduce the case of a general M admitting a smooth C^∞ action of $\mathbb{T}$ to the case of $M = I^{d-1} \times \mathbb{T}$, where $I = [0, 1]$ is the standard unit interval, with $S_\theta : I^{d-1} \times \mathbb{T} \to I^{d-1} \times \mathbb{T}$ given by

$$S_\theta(x_1, \dots, x_d) = (x_1, \dots, x_{d-1}, x_d + \theta \bmod 1).$$

Let σ denote the effective $\mathbb{T}$ action on M. For $q \geq 1$ we denote by F_q the set of fixed points of the map $\sigma(1/q, \cdot)$ and let $B := \partial M \cup \bigcup_{q \geq 1} F_q$ be the set of exceptional points.

We quote the following proposition of [**2**] that is similar to other statements in [**1, 6**]

PROPOSITION 1. *[**2**, proposition 5.2] Let M be an d-dimensional compact connected C^∞ manifold with an effective circle action σ preserving a smooth volume μ. Then here exists a continuous surjective map $\Gamma : I^{d-1} \times \mathbb{T} \to M$ with the following properties*

1. *The restriction of Γ to $(0, 1)^{d-1} \times \mathbb{T}$ is a C^∞ diffeomorphic embedding;*
2. *$\mu(\Gamma(\partial(I^{d-1} \times \mathbb{T})) = 0$;*
3. *$\Gamma(\partial(I^{d-1} \times \mathbb{T})) \supset B$;*
4. *$\Gamma_*(\lambda) = \mu$;*
5. *$\sigma\,\Gamma = \Gamma\, S$.*

An application of Proposition 1 at each step allows us to conclude Theorem 1 from the special case $M = I^{d-1} \times \mathbb{T}$. Thus the construction need only be carried out for two specific manifolds; $M = \mathbb{T}^d$ or $M = I^{d-1} \times \mathbb{T}$. For both we take the action $S_\theta : M \to M$ given by

$$S_\theta(x_1, \dots, x_d) = (x_1, \dots, x_{d-1}, x_d + \theta \bmod 1)$$

that preserves the smooth unit volume λ induced by the usual Lebesgue measure on $\mathbb{R}^d$.

2.3. *Partitions and Measure-Theoretic Isomorphism* The most difficult property to define on a finite scale is that of measure-theoretic isomorphism to a circle rotation. We use the abstract theory of Lebesgue spaces. Given an isomorphism of measures space $(M_1, \mathfrak{B}_1, \mu_1)$ and $(M_2, \mathfrak{B}_2, \mu_2)$ there is a natural isomorphism of the associated measure-algebras. If both the measure-spaces are Lebesgue spaces then the converse is true; every isomorphism of the measure-algebras arises from a point isomorphism of the measure spaces. This is the crucial observation that leads to the following abstract lemma, which appears as [**1**, Lemma 4.1].

Given a partition ξ of a space M we write $\xi(x)$ for the atom of the partition which contains x. We say that a sequence of partitions ξ_n generates if there is a set F of full measure such that for every $x \in F$ we have

$$\{x\} = F \cap \bigcap_{n=1}^{\infty} \xi_n(x).$$

LEMMA 1. *Let M_1 and M_2 be Lebesgue spaces. Let $(\xi_n^{(i)})_{n=1}^{\infty}$ be a monotone sequence of finite measurable partitions of M_i that generates. Let $(T_n^{(i)})_{n=1}^{\infty}$ be a sequence of automorphisms of M_i such that*

1. *$(T_n^{(i)})_{n=1}^{\infty}$ converges in the weak topology to an automorphism $T^{(i)}$ of M_i.*
2. *$T_n^{(i)} \xi_n^{(i)} = \xi_n^{(i)}$.*

Suppose that for each n there exists a measure-theoretic isomorphism K_n : $M_1/\xi_n^{(1)} \to M_2/\xi_n^{(2)}$ of the probability vectors such that:

3. *$K_n^{-1}\, T_n^{(2)}\Big|_{\xi_n^{(2)}} K_n = T_n^{(1)}\Big|_{\xi_n^{(1)}}$.*
4. *for all $\Delta \in \xi_{n-1}^{(1)}$*
$$K_n \Delta = K_{n-1}\Delta.$$

Then the automorphisms $T^{(1)}$ and $T^{(2)}$ are measure-theoretically isomorphic.

Consider the partition of $\mathbb{T}$ given by

$$\tilde{\eta}_q := \{\tilde{\Delta}_{i,q} : 0 \le i < q^d\} \tag{4}$$

where $\tilde{\Delta}_{i,q} := [iq^{-d}, (i+1)q^{-d})$. This partition is preserved under the action $R_{p/q}$. For any increasing sequence of q_n the sequence of partitions $\tilde{\eta}_{q_n}$ generates. Let $M_2 = \mathbb{T}$, $\xi_n^{(2)} = \tilde{\eta}_{q_n}$ and $T_n^{(2)} = R_{\alpha_n}$. Since q_n divides q_{n+1} we have $\tilde{\eta}_{q_n} < \tilde{\eta}_{q_{n+1}}$.

Let $\pi_d : M \to \mathbb{T}$ denote the projection onto the last component of M. We obtain a partition of M by

$$\eta_q = \pi_d^{-1}\tilde{\eta}_q = \{\Delta_{i,q} : 0 \le i < q^d\} \tag{5}$$

where

$$\Delta_{i,q} := \{x : x_d \in [iq^{-d}, (i+1)q^{-d})\},$$

see Figure 1. Since $\pi_d\, S_\alpha = R_\alpha\, \pi_d$ the partition η_q is preserved under the action of $S_{p/q}$ and, moreover, the action of $S_{p/q}$ on η_q is conjugated with that of $R_{p/q}$ on $\tilde{\eta}_q$. Unfortunately the sequence of partitions η_{q_n} does not generate.

Let $M_1 = M$ and define the sequence of partitions

$$\xi_n^{(1)} := H_{n+1}\eta_{q_n} = H_n h_{n,q_n} \eta_{q_n}. \tag{6}$$

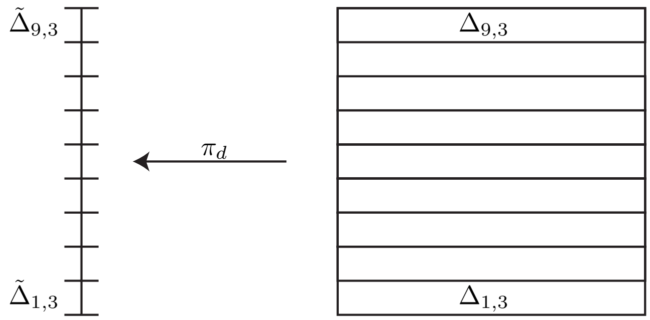

FIGURE 1. The partition η_3 of either $I \times \mathbb{T}$ or $\mathbb{T}^2$ and the partition $\tilde{\eta}_3$ of $\mathbb{T}$.

Unlike the sequence η_{q_n}, the sequence $\xi_n^{(1)}$ can be made to generate. We construct $h_{n,q}$ as a diffeomorphism of $\pi_d^{-1}[0, q^{-1}]$ and extend it to all of M by requiring that it commute with $S_{q^{-1}}$. Then

1. Since q_{n-1}^d divides q_n we have for $0 \le i < q_{n-1}^d$
$$h_{n,q_n}\Delta_{i,q_{n-1}} = \Delta_{i,q_{n-1}}.$$
2. Since q'_n divides q_n we have
$$h_{n,q_n}\, S_{\alpha_n} = S_{\alpha_n}\, h_{n,q_n}.$$

As $\eta_{q_{n-1}} < \eta_{q_n}$ we have $H_{n+1}\eta_{q_{n-1}} < H_{n+1}\eta_{q_n}$. By the first of our two properties we have that $H_{n+1}\eta_{q_{n-1}} = H_n\eta_{q_{n-1}}$ and hence $\xi_{n-1}^{(1)} < \xi_n^{(1)}$. Thus $\{\xi_n^{(1)}\}$ is a monotone sequence of partitions as required by Lemma 1. The second property ensures that $T_n\xi_n^{(1)} = \xi_n^{(1)}$. Define the map
$$K_n = \pi_d\, H_{n+1}^{-1}.$$
Using the two properties we have that
$$K_n\, T_n^{(1)} = T_n^{(2)}\, K_n$$
$$K_n(H_n\Delta_{i,q_{n-1}}) = K_{n-1}(H_n\Delta_{i,q_{n-1}})$$
as required by Lemma 1.

This completes the proof of the main theorem except for the proof that the sequence T_n converges in $\text{Diff}^\infty(M, \lambda)$ and the proof that $\xi_n^{(1)}$ generates.

2.4. *Construction of the Conjugating Maps.* We will carry out the constructions for $M = \mathbb{T}^d$ and $M = I^{d-1} \times \mathbb{T}$ simultaneously. The proof of unique ergodicity in the case $M = \mathbb{T}^d$ will appear in a later section.

LEMMA 2. *Let $n > 2d$ and $q \in N$. There exists a map $h_{n,q} \in \text{Diff}^\infty(M, \lambda)$ and a set $E_{n,q} \subset M$ such that:*

1. $h_{n,q}S_{q^{-1}} = S_{q^{-1}}h_{n,q}$ *and* $h_{n,q}\big(\pi_d^{-1}[0, q^{-1}]\big) = \pi_d^{-1}[0, q^{-1}]$.
2. $\lambda(E_{n,q}) > 1 - 4\frac{d-1}{n^2}$.
3. *for each* $0 \le i < q^d$,
$$\text{diam}\, h_{n,q}(\Delta_{i,q} \cap E_{n,q}) < \sqrt{d}q^{-1}.$$

2.4.1. *Heuristic Construction.* In order to motivate the construction of the family of conjugacy maps we first construct a family of measure-preserving discontinuous maps $\tilde{h}_q$ such that $\tilde{h}_q$ commutes with $S_{q^{-1}}$ and carries each $\Delta_{i,q}$ into a d-dimensional cube with side-length q^{-1}.

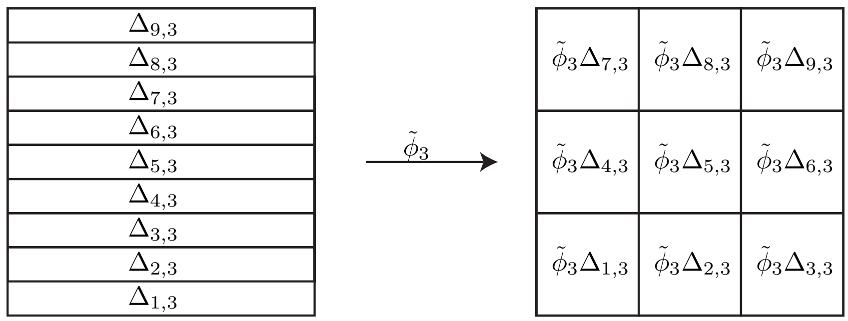

FIGURE 2. Action of $\tilde{\phi}_3 = \tilde{h}_3$ on the partition η_3.

Let $\tilde{\phi}_q$ be defined on $[0,1] \times [0,q^{-1}]$ by letting it act on the interior by

$$\tilde{\phi}_q(x,y) := (qy, q^{-1}(1-x))$$

and extend it to all of $[0,1]\times[0,1]$ by requiring $\tilde{\phi}_q(x, y+q^{-1}) = \tilde{\phi}_q(x,y) + (0,q^{-1})$. Define $\tilde{\phi}_q^{(i)}$ by

$$[\tilde{\phi}_q^{(i)}]_j(x_1,\dots,x_d) = \begin{cases} [\tilde{\phi}_q]_1(x_i,x_{i+1}) & j=i \\ [\tilde{\phi}_q]_2(x_i,x_{i+1}) & j=i+1 \\ x_j & \text{otherwise} \end{cases} \tag{7}$$

The map $\tilde{h}_q$ is defined by

$$\tilde{h}_q := \tilde{\phi}_q^{(1)} \cdots \tilde{\phi}_q^{(d-1)}.$$

Each $\Delta_{i,q}$ is mapped, by $\tilde{h}_q$, into a cube of side-length q^{-1}. The map $\tilde{h}_q$ commutes with $S_{q^{-1}}$ since $\tilde{\phi}_q^{(d-1)}$ commutes with $S_{q^{-1}}$ by construction and the other $\tilde{\phi}_q^{(i)}$ do not affect x_d, see Figure 2.

2.4.2. *Proof of Lemma 2* Our family of conjugating maps $h_{n,q}$ is constructed using the same process as $\tilde{h}_q$ above. Clearly control of some of the space must be relinquished in order to be able to produce a C^∞ volume preserving map. One additional complication arises ensuring that we retain sufficient control over every orbit. Let φ_n denote a C^∞ map of the unit square satisfying

1. $\varphi_n = \mathrm{Id}$ on a neighborhood of the boundary.
2. φ_n acts as a pure rotation by $\frac{\pi}{2}$ on

$$\square_n = \Big[\frac{1}{n^2}, 1-\frac{1}{n^2}\Big] \times \Big[\frac{1}{n^2}, 1-\frac{1}{n^2}\Big].$$

3. φ_n preserves Lebesgue measure.

To construct such a map we observe that the unit square can be mapped to the unit circle by

$$A(x,y) = \Big((2x-1)\sqrt{1-\frac{(2y-1)^2}{2}}, (2y-1)\sqrt{1-\frac{(2x-1)^2}{2}}\Big).$$

This map is continuous and is a diffeomorphism on the interior. Moreover the map is equivariant under rotation by $\frac{\pi}{2}$. The image of $\square_n$ under A is compact and hence we can find radii $r_1 < r_3 < 1$ such that the image is contained in the interior of the disk of radius r_1. We consider the restriction of the measure $A_*\lambda$ to the disk of radius r_3. This is a smooth manifold with boundary endowed with a smooth measure. Using polar coordinates and the method of [**1**, Theorem 1.2] we can find a radius r_2 with $r_1 < r_2 < r_3$ and a diffeomorphism $S(r,\theta) = (g(r,\theta),\theta)$ such that

$$S_*A_*\lambda = \begin{cases} A_*\lambda & r \le r_1 \\ \lambda & r_2 \le r \le r_3. \end{cases}$$

Since the measure $A_*\lambda$ is invariant under rotation by $\frac{\pi}{2}$ we may choose g so that that $g(r,\theta+\frac{\pi}{2}) = g(r,\theta)$ and thus S commutes with rotation by $\frac{\pi}{2}$. Thus the measure $S_*A_*\lambda$ is invariant under rotation by $\frac{\pi}{2}$. Let $\rho : [0,r_3] \to [0,\frac{\pi}{2}]$ be a C^∞ smooth function that is 0 is a neighborhood of r_3 and $\frac{\pi}{2}$ on $[0,r_2]$. Now we consider the smooth map on the disk of radius r_3 defined by

$$B(r,\theta) = (r, \theta + \rho(r))$$

Since the measure on the disk of radius r_2 is invariant under rotation by multiples of $\frac{\pi}{2}$, and the measure on the annulus $r_2 \le r \le r_3$ is Lebesgue and therefore invariant under all rotations, this map is measure preserving. Since $S^{-1}\,B\,S$ is identity in a neighborhood of the boundary of the disk of radius r_3 we can extend it by identity to a map of the whole unit disk. Now $\varphi_n = A^{-1}\,S^{-1}\,B\,S\,A$ is the required map of the square. By symmetry of φ_n we see that $|||\varphi_n|||_k = |||\varphi_n^{-1}|||_k$.

Let $C_q(x,y) := (x, q^{-1}y)$ and define $\phi_{n,q}$ on $[0,1]\times[0,q^{-1}]$ by

$$\phi_{n,q} := C_q \varphi_n C_q^{-1}. \tag{8}$$

Extend $\phi_{n,q}$ to the entire unit square by requiring that

$$\phi_{n,q}(x, y+q^{-1}) = \phi_{n,q}(x,y) + (0, q^{-1}).$$

This agrees with $\tilde{\phi}_q$ on a set of volume $(1-2/n^2)^2$ which we estimate from below by $1-4/n^2$. Analogously to our earlier definition of $\tilde{\phi}_q^{(i)}$ we define $\phi_{n,q}^{(i)}$.

$$[\phi_{n,q}^{(i)}]_j(x_1,\dots,x_d) = \begin{cases} [\phi_{n,q}]_1(x_i,x_{i+1}) & j = i \\ [\phi_{n,q}]_2(x_i,x_{i+1}) & j = i+1 \\ x_j & \text{otherwise} \end{cases}$$

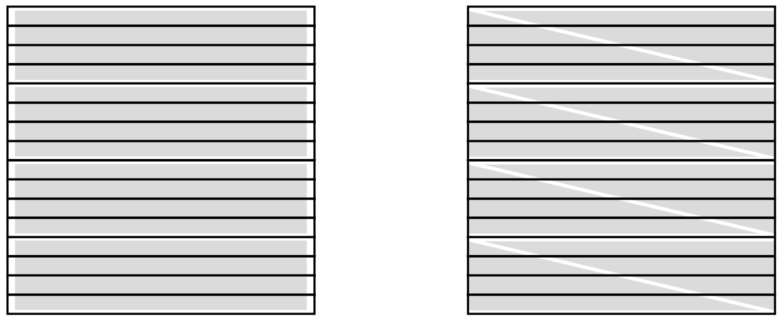

FIGURE 3. The set E_{n_q} for the case $M = I \times \mathbb{T}$ (left) and for the case $M = \mathbb{T}^2$ (right).

2.4.3. *$M = I^{d-1} \times \mathbb{T}$ Case.* We define the conjugating map $h_{n,q} : I^{d-1} \times \mathbb{T} \to I^{d-1} \times \mathbb{T}$ by

$$h_{n,q} := \phi^{(1)}_{n,q} \cdots \phi^{(d-1)}_{n,q}.$$

This map agrees with $\tilde{h}_q$ on a set $E_{n,q}$ given by

$$E^c_{n,q} = \bigcup_{i=1}^{d-1} \pi_i^{-1}\Big(\big[0, \frac{1}{n^2}\big) \cup \big(1 - \frac{1}{n^2}, 1\big]\Big) \cup \bigcup_{j=1}^{d-1} \bigcup_{k=1}^{q^j} \pi_d^{-1}\big(\frac{k}{q^j} - \frac{1}{n^2 q^j}, \frac{k}{q^j} + \frac{1}{n^2 q^j}\big), \quad (9)$$

see Figure 2.4.4. Treating the sets on the right as disjoint we can estimate

$$\lambda(E_{n,q}) > 1 - 4\frac{d-1}{n^2}. \quad (10)$$

2.4.4. *$M = \mathbb{T}^d$ Case.* In order to produce a unique ergodic diffeomorphism T it is necessary to control *all* orbits. The set $E_{n,q}$ constructed above for the case of $M = I^{d-1} \times \mathbb{T}$ excludes entire orbits. In order to rectify this requires one more map. Let $\psi_q : \mathbb{T}^d \to \mathbb{T}^d$ denote the translation

$$\psi_q(x_1, \ldots, x_{d-1}, x_d) := (x_1, \ldots, x_{d-1}, x_d) + x_d(q, \ldots, q, 0) \mod 1. \quad (11)$$

Obviously ψ_q commutes with $S_{q^{-1}}$ and preserves the Lebesgue measure. Furthermore, since ψ_q does not affect the last coordinate, it preserves each $\Delta_{i,q}$. For the uniquely ergodic case we define

$$h_{n,q} := \phi^{(1)}_{n,q} \cdots \phi^{(d-1)}_{n,q} \psi_q \quad (12)$$

Exactly as for the ergodic case $h_{n,q}$ agrees with $\tilde{h}_q$ on a set $E_{n,q}$ with

$$\lambda(E_{n,q}) > 1 - 4\frac{d-1}{n^2}.$$

The map ψ_q ensures that $E_{n,q}$ contains most of *every* orbit, see Figure 2.4.4.

2.5. *Analytic Properties*

2.5.1. *Notation* All of our diffeomorphisms $h : I^{d-1} \times \mathbb{T} \to I^{d-1} \times \mathbb{T}$ are the identity in a neighborhood of the boundary and hence can be identified with a diffeomorphism $h : \mathbb{T}^d \to \mathbb{T}^d$. Defining a topology on $\mathrm{Diff}^k(\mathbb{T}^d, \mathbb{T}^d)$ defines a topology on the closure of the space of diffeomorphisms $h : I^{d-1} \times \mathbb{T} \to I^{d-1} \times \mathbb{T}$ that are the identity in a neighborhood of the boundary.

Let $f, g \in C^0(\mathbb{T}^d, \mathbb{T}^d)$. We define

$$\hat{d}_0(f, g) = \max_{x \in M} d\big(f(x), g(x)\big).$$

Let $f \in C^k(\mathbb{R}^d, \mathbb{R})$. Given $a \in \mathbb{N}^d$ we denote $|a| := a_1 + \cdots + a_d$ and

$$D_a f := \frac{\partial^{|a|} f}{\partial x_1^{a_1} \ldots \partial x_d^{a_d}}.$$

Using this we can define

$$|\!|\!| f |\!|\!|_k = \max_{1 \le |a| \le k} \max_{x \in M} |D_a f(x)|.$$

For $f \in C^k(\mathbb{R}^d, \mathbb{R}^d)$ we define

$$|\!|\!| f |\!|\!|_k = \max_{1 \le i \le d} \max_{1 \le |a| \le k} \max_{x \in M} |D_a f_i(x)|.$$

For $h : \mathbb{T}^d \to \mathbb{T}^d$ we can define a natural lift $\hat{h} : \mathbb{R}^d \to \mathbb{R}^d$. Now given $f, g \in C^k(\mathbb{T}^d, \mathbb{T}^d)$ we define

$$\hat{d}_k(f, g) = \max\{d_0(f, g), |\!|\!| \hat{f} - \hat{g} |\!|\!|_k\}$$

Finally, for $f, g \in \mathrm{Diff}^k(\mathbb{T}^d, \mathbb{T}^d)$ we define

$$d_k(f, g) = \max\{\hat{d}_k(f, g), \hat{d}_k(f^{-1}, g^{-1})\}$$

The metric defined in this way is equivalent to the usual one defined via the operator norms but is easier to work with for explicit estimates. For further details consult [**5**].

2.5.2. *Estimates*

LEMMA 3. *We have the following estimate:*

$$|\!|\!| h_{n,q} |\!|\!|_k < C_1 q^{dk} \qquad |\!|\!| h_{n,q}^{-1} |\!|\!|_k < C_1 q^{dk} \tag{13}$$

where C_1 depends on d, k, and n but is independent of q.

Proof: By direct computation we obtain

$$|\!|\!| \phi_{n,q}^{(i)} |\!|\!|_k < q^k |\!|\!| \varphi_n |\!|\!|_k, \qquad |\!|\!| (\phi_{n,q}^{(i)})^{-1} |\!|\!|_k < q^k |\!|\!| \varphi_n^{-1} |\!|\!|_k \tag{14}$$

and

$$|\!|\!| \psi_q |\!|\!|_k < q, \qquad |\!|\!| \psi_q^{-1} |\!|\!|_k < q. \tag{15}$$

We claim that partial derivatives with $|a| = k$ consist of sums of products of at most $(d-1)k$ terms of the form

$$\big(D_b[\phi_{n,q}^{(i)}]_j\big)(\phi_{n,q}^{(i+1)} \dots \phi_{n,q}^{(d-1)}\psi_q) \tag{16}$$

with $|b| \leq k$ and at most k terms of the form

$$D_c[\psi_q]_j \tag{17}$$

with $|c| = 1$. This is true for $|a| = 1$ by computation and, by the product and chain rules, if it is true for $|a| = k$ then it is true for $|a| = k+1$. By induction it is therefore true for all k.

Now suppose that each summand in the expression for $D_a[h_{n,q}]_j$ satisfies an estimate of the form (13) for $1 \leq j \leq d$ and for $|a| = k$. We will then show that each summand in the expression for $D_a[h_{n,q}]_j$ for $1 \leq j \leq d$ and $|a| = k+1$ satisfies an estimate of the form (13). The estimate (13) for follows from the estimate on the summands.

We use our structure theorem for k. Differentiating a term of the form (16) we get a sum of products of $d+1-i$ terms. The first is of the form (16) but with the power of the derivative raised by 1. The next $d-1-i$ terms are first partial derivatives of $\phi_{n,q}^{(i+1)}, \dots, \phi_{n,q}^{(d-1)}$. The final term is a first partial derivative of ψ_q. Applying the estimates (14) we see that the required power of q has been increased by at most d. Differentiating (17) gives zero since ψ_q is linear.

The same considerations give us the estimate for $h_{n,q}^{-1}$ since it has the same structure and we have the same estimates on the constituent maps.

□

By an application of Faà di Bruno's formula we obtain the following corollary.

Corollary 1. *We have the following estimate*

$$\|\!|H_n h_{n,q}|\!\|_k < C_2 q^{kd} \qquad \|\!|h_{n,q}^{-1} H_n^{-1}|\!\|_k < C_2 q^{kd} \tag{18}$$

where C_2 depends on d, H_n, n, and k but is independent of q.

2.6. *Completing the Construction* Having now constructed the family of maps $h_{n,q}$ from which the maps H_n are assembled it remains only to explain how we choose the sequence q_n. The choice of q_n determines α_n as the best approximation to α with denominator q_n.The choices of $q_1, ..., q_{n-1}$ completely determines H_n. We show how given H_n we choose q_n so that T_n has the desired properties.

In the original Anosov and Katok method of construction the choice of α_n in the definition of T_n (2) determined the distance between the already determined T_{n-1} and T_n in Diff^n. The observation there was that if α_n could be chosen arbitrarily close to α_{n-1} then the transformation T_n could be made arbitrarily close to T_{n-1}. The advantages of this approach are that no estimates on the maps H_n are required. Unfortunately this approach is inconsistent with ensuring that the sequence α_n converges to an a priori given number α. In the approach we take the choice of q_n (and hence of α_n) determines the distance between T_n and, the as-yet undetermined transformation, T_{n+1}. Since the choice of q_n fixes the conjugacy map H_{n+1} the only undetermined quantity in T_{n+1} is the choice of α_{n+1}. Supposing only that the choice of α_{n+1} will be a better approximation to α than α_n we are able to estimate the distance between T_n and T_{n+1} knowing only the choice of α_n.

LEMMA 4. *Let $k \in \mathbb{N}$. For all $h \in \mathrm{Diff}^{k+1}(M)$ and all $\alpha, \beta \in \mathbb{R}$ we obtain*

$$d_k(h\, S_\alpha\, h^{-1}, h\, S_\beta\, h^{-1}) \leq C_3 \, \max\{\|h\|_{k+1}^{k+1}, \|h^{-1}\|_{k+1}^{k+1}\}\, |\alpha - \beta|$$

where C_3 depends only on k.

This estimate is wasteful. It ignores the trade-off between the order of the derivatives that appear and their number. Since we just have to control derivatives on a polynomial size the estimate is sufficient.

Proof: For $k = 0$ we have the estimate

$$d_0(h\, S_\alpha\, h^{-1}, h\, S_\beta\, h^{-1}) \leq \|h\|_1 |\alpha - \beta|$$

by the mean value theorem. We claim that for $a \in \mathbb{N}^d$ with $|a| = k$ the partial derivative

$$D_a[h_i\, S_\alpha\, h^{-1} - h_i\, S_\beta\, h^{-1}]$$

will consist of a sum of terms with each term being the product of a single partial derivative

$$\big(D_b h_i\big)(S_\alpha h^{-1}) - \big(D_b h_i\big)(S_\beta h^{-1}) \tag{19}$$

with $|b| \leq k$, and at most k partial derivatives of the form

$$D_b h_j^{-1} \tag{20}$$

with $|b| \leq k$. For $k = 1$ we have

$$\frac{\partial}{\partial x_j}[h_i\, S_\alpha\, h^{-1} - h_i\, S_\beta\, h^{-1}] = \sum_{l=1}^{d} \Big(\frac{\partial h_i}{\partial x_l}\, S_\alpha\, h^{-1} - \frac{\partial h_i}{\partial x_l}\, S_\beta\, h^{-1}\Big) \frac{\partial h_l^{-1}}{\partial x_j}.$$

We proceed by induction. By the product rule we need only consider the effect of differentiating (19) and (20). Differentiating (19) with respect to x_j we obtain

$$\sum_{l=1}^{d} \Big(\frac{\partial D_b h_i}{\partial x_l}\, S_\alpha\, h^{-1} - \frac{\partial D_b h_i}{\partial x_l}\, S_\beta\, h^{-1}\Big) \frac{\partial h_l^{-1}}{\partial x_j}.$$

which increases the number of terms of the form (20) by 1. Differentiating (20) we get another term of the form (20) but with $|b| \leq k + 1$.

We estimate

$$\|D_a h_i\, S_\alpha\, h^{-1} - D_a h_i\, S_\beta\, h^{-1}\|_0 \leq \|h\|_{|a|+1} |\alpha - \beta|$$
$$\|D_a h_l^{-1}\|_0 \leq \|h\|_{|a|}$$

These estimates together with claimed structure of the partial derivatives, and the fact that the inverse maps have the same structure, completes the proof. The constant C_3 is the number of terms in the sum which depends only on k and not on the map h. □

Define $F_n := H_{n+1}(E_{n,q_n})$ and let $F := \liminf_{n\to\infty} F_n$. The Borel-Cantelli Lemma states that if $\sum_{n=1}^{\infty} \mu(F_n^c) < \infty$ then $\mu(F) = \mu(\liminf_{n\to\infty} F_n) = 1$. From Lemma 2 we have

$$\sum_{n=1}^{\infty} \mu(F_n^c) < \sum_{n=1}^{\infty} \frac{4(d-1)}{n^2} < \infty$$

so by the Borel-Cantelli Lemma $\mu(F) = 1$. We will show that any point in F has a unique coding relative to the sequence of partitions ξ_n.

PROPOSITION 2. *Let ϵ_n be a summable sequence of positive numbers. There is a choice of $\{q'_n\}$ such the transformations T_n defined by* (2) *satisfy*

1. $d_n(T_n, T_{n+1}) < \epsilon_n$.

2. *for* $A \in \xi_n$

$$\mathrm{diam}(A \cap F_n) < \epsilon_n$$

Proof: By the definition of a Liouville number for any $C > 0$ and $m \in \mathbb{N}$ we can find $q'_n > q_{n-1}$ such that $\alpha_n := p'_n/q'_n$ is a better approximation to α than α_{n-1} and such

$$C\,(q'_n)^m \left|\frac{p'_n}{q'_n} - \alpha\right| < \epsilon_n$$

Recall that we define $q_n = q_{n-1}^d q'_n$. Since $q_n < (q'_n)^{d+1}$ we have that for any $C > 0$ and $m \in \mathbb{N}$ we can find find q'_n such that $\alpha_n := p'_n/q'_n$ is a better approximation to α than α_{n-1} and such

$$C\,q_n^m \left|\frac{p'_n}{q'_n} - \alpha\right| < \epsilon_n$$

Now combining (18) and Lemma 4 there exists $C > 0$ and $m \in \mathbb{N}$ such that

$$\begin{aligned} d_n(T_n, T_{n+1}) &< C\,q_n^m\,|\alpha_n - \alpha_{n+1}| \\ &< 2C\,q_n^m\,|\alpha_n - \alpha|. \end{aligned}$$

This is the crucial estimate for us, it enable us to capture all Liouville rotations. In [**1**] no such estimate exists and convergence is guaranteed by making $|\alpha_n - \alpha_{n+1}|$ arbitrarily small. This would not be compatible with convergence to α.

Similarly for $H_{n+1}\Delta_{i,q_n} \in \xi_n$ we have

$$\begin{aligned} \mathrm{diam}(H_{n+1}\Delta_{i,q_n} \cap F_n) &= \mathrm{diam}(H_n h_{n,q_n}(\Delta_{i,q_n} \cap E_{n,q_n})) \\ &\le \|H_n\|_1 \,\mathrm{diam}\, h_{n,q_n}(\Delta_{i,q_n} \cap E_{n,q_n}) \\ &\le \|H_n\|_1 \sqrt{d} q_n^{-1} \end{aligned}$$

using Lemma 2. Similar estimates appear in [**1**]. This depends only on the size of q_n which can be chosen arbitrarily large. Thus we see that we can choose α_n such that the required two properties hold. □

Since ϵ_n is summable we have that $\{T_n\}$ is a Cauchy sequence in $\mathrm{Diff}^\infty(M, \lambda)$ and hence converges to some $T \in \mathrm{Diff}^\infty(M, \lambda)$. For any $x \in F$ we have $x \in F_n$ for all but finitely many n. Thus, by Proposition 2, we have for all $x \in F$

$$\bigcap_{n=1}^{\infty} \xi_n(x) \cap F = \{x\}.$$

This shows that $\{\xi_n\}$ is a generating partition and hence completes the proof of Theorem 1.

3. *Unique Ergodicity*

When $M = \mathbb{T}^d$ we wish to prove unique ergodicity. We will use the following abstract lemma, also used in [**6**].

LEMMA 5. *Let q_n be an increasing sequence of natural numbers and $T_n : X \to X$ a sequence of transformations which converge uniformly to a transformation T. Suppose that for each continuous function φ from a dense set of continuous functions Φ there is a constant c such that*

$$\frac{1}{q_n}\sum_{i=0}^{q_n-1}\varphi(T_n^i x) \xrightarrow[n\to\infty]{} c \text{ uniformly} \tag{21}$$

and

$$d^{(q_n)}(T_n, T) := \max_x \max_{0\le i<q_n} d(T_n^i x, T^i x) \to 0 \tag{22}$$

Then T is uniquely ergodic

Proof: Condition (22) implies that

$$\|\frac{1}{q_n}\sum_{i=0}^{q_n-1}\varphi(T_n x) - \frac{1}{q_n}\sum_{i=0}^{q_n-1}\varphi(Tx)\|_0 \to 0$$

and then condition (21) becomes the standard result that if the Birkhoff sums converge uniformly then the map is uniquely ergodic [**4**]. □

To establish condition (21) it is insufficient to know only that $E_{n,q}$ has large measure, we also need to know that most of every S_θ orbit intersects $E_{n,q}$.

For each $x \in \mathbb{T}^d$ define $\sigma_x : \mathbb{T} \to \mathbb{T}^d$ by $\sigma_x\theta = S_\theta x$.

LEMMA 6. *Let $q > dn^2$. For each $x \in \mathbb{T}^d$ there is a set $J_{n,q}^{(x)} \subset \mathbb{T}^d$, measurable with respect to η_q, with measure*

$$\lambda(J_{n,q}^{(x)}) > 1 - \frac{4d}{n^2} \tag{23}$$

such that if $\Delta_{i,q} \subset J_{n,q}^{(x)}$ then

$$\sigma_x^{-1}(\Delta_{i,q} \cap E_{n,q}^c) = \varnothing, \tag{24}$$

$$\lambda(\Delta_{i,q} \cap E_{n,q}) > \big(1 - \frac{2(d-1)}{n^2}\big)\lambda(\Delta_{i,q}). \tag{25}$$

Proof: It is immediate that

$$(E'_{n,q})^c = \bigcup_{i=1}^{d-1}\pi_i^{-1}\big(-\frac{1}{n^2}, \frac{1}{n^2}\big) \cup \bigcup_{j=1}^{d-1}\bigcup_{k=1}^{q^j}\pi_d^{-1}\big(\frac{k}{q^j} - \frac{1}{n^2 q^j}, \frac{k}{q^j} + \frac{1}{n^2 q^j}\big) \tag{26}$$

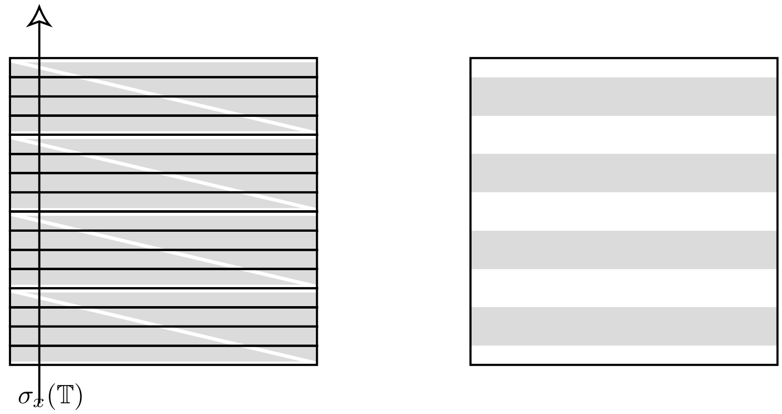

FIGURE 4. The orbit of $x \in \mathbb{T}^2$, indicated by the arrow on the left, combines with $E_{n,q}$, indicated by the shaded region on the left, to produce the set $J^{(x)}_{n,q}$, indicated by the shaded region on the right.

Let x be arbitrary. We compute $\sigma_x^{-1}\psi_q(E'_{n,q})^c$ using (26) and (11).

$$\sigma_x^{-1}\psi_q^{-1}\pi_i^{-1}\big(-\frac{1}{n^2},\frac{1}{n^2}\big) = \bigcup_{l=1}^{q}\Big(\frac{l}{q}-\frac{1}{n^2q}-x_d-\frac{x_i}{q},\frac{l}{q}+\frac{1}{n^2q}-x_d-\frac{x_i}{q}\Big)$$

$$\sigma_x^{-1}\psi_q^{-1}\pi_d^{-1}\big(\frac{k}{q^j}-\frac{1}{n^2q^j},\frac{k}{q^j}+\frac{1}{n^2q^j}\big) = \big(\frac{k}{q^j}-\frac{1}{n^2q^j}-x_d,\frac{k}{q^j}+\frac{1}{n^2q^j}-x_d\big)$$

This excluded set of τ consists of at most $(d-1)q+q^{d-1}$ intervals. Expanding these intervals to make them measurable with respect to $\sigma_x^{-1}\eta_q$ excludes an additional set of measure at most

$$\frac{2}{q^d}\big((d-1)q+q^{d-1}\big) < \frac{4}{n^2}.$$

Let E denote the smallest set that is measurable with respect to the algebra generated by the partition $\sigma_x^{-1}\eta_q$ and contains $\sigma_x^{-1}E^c_{n,q}$. We have $\lambda(E) = 4d/n^2$. Define the set $J^{(x)}_{n,q}$ to be the η_q measurable set satisfying

$$\sigma_x^{-1}J^{(x)}_{n,q} = E^c,$$

see Figure 4. □

Note that the proportion in (23) is lower than the proportion in (10). We have had to give up control over parts of each orbit in order to gain control over all orbits. The set $J^{(x)}_{n,q}$ consists of those atoms of η_q where we have control over the behavior of all of $S_\theta x$ under $h_{n,q}$.

Using the geometric information contained in these lemmas we can prove a distribution result.

PROPOSITION 3. *Let $\epsilon > 0$, $q \in \mathbb{N}$, and φ be a $(\sqrt{d}q^{-d},\epsilon)$-uniformly continuous function, i.e*

$$\varphi(B_{\sqrt{d}q^{-d}}(x)) \subset B_\epsilon(\varphi(x)).$$

For all $q' \in \mathbb{N}$ *and for all* $x \in \mathbb{T}^d$,

$$\left| \frac{1}{q'} \sum_{i=0}^{q'-1} \varphi(h_{n,q} S^i_{1/q'} x) - \int \varphi d\lambda \right| < \frac{14d}{n^2} \|\varphi\|_0 + \frac{2q^d}{q'} \|\varphi\|_0 + 2\epsilon. \tag{27}$$

Proof: For $x, y \in \Delta_{i,q} \cap E_{n,q}$ we have

$$d(h_{n,q}x, h_{n,q}y) \leq \operatorname{diam} h_{n,q}(\Delta_{i,q} \cap E_{n,q}) \leq \sqrt{d} q^{-d}.$$

By the hypothesis on φ we have $|\varphi(h_{n,q}x) - \varphi(h_{n,q}y)| < 2\epsilon$. Averaging over all $y \in \Delta_{i,q} \cap E_{n,q}$ we obtain for any $x \in \Delta_{i,q} \cap E_{n,q}$,

$$\left| \varphi(h_{n,q}x) - \frac{1}{\lambda(\Delta_{i,q} \cap E_{n,q})} \int_{h_{n,q}(\Delta_{i,q} \cap E_{n,q})} \varphi \, d\lambda \right| < 2\epsilon. \tag{28}$$

Let $\mathcal{O}^{(x)}$ consist of $\lfloor \frac{q'}{q^d} \rfloor q^d$ points of the orbit of x under $S_{1/q'}$ that are equidistributed among the atoms of the partition η_q. There are at most q^d exceptional points outside of $\mathcal{O}^{(x)}$.

By (24) for $\Delta_{i,q} \subset J^{(x)}_{n,q}$ the number of points from $\mathcal{O}^{(x)}$ in $\Delta_{i,q} \cap E_{n,q}$ is $\lfloor \frac{q'}{q^d} \rfloor$. Let $I := \{0 \leq i < q' : S^i_{1/q'} x \in J^{(x)}_{n,q} \cap \mathcal{O}^{(x)}\}$ be the equidistributed points in good atoms. Using this count and (28) we obtain

$$\left| \frac{1}{q'} \sum_{i \in I} \varphi(h_{n,q} S^i_{1/q'} x) \right.$$
$$\left. - \frac{1}{q'} \sum_{\Delta_{i,q} \subset J^{(x)}_{n,q}} \left\lfloor \frac{q'}{q^d} \right\rfloor \frac{1}{\lambda(\Delta_{i,q} \cap E_{n,q})} \int_{h_{n,q}(\Delta_{i,q} \cap E_{n,q})} \varphi \, d\lambda \right| < 2\epsilon.$$

The remaining estimates just formalize the observation that since $J^{(x)}_{n,q}$ is nearly full measure and since I is nearly all of the orbit the above estimate implies (27).

Since there are at most q^d points in the orbit of $S_{1/q'}$ but outside $\mathcal{O}^{(x)}$ we obtain

$$\left| \frac{1}{q'} \sum_{\Delta_{i,q} \subset J^{(x)}_{n,q}} \left\lfloor \frac{q'}{q^d} \right\rfloor \frac{1}{\lambda(\Delta_{i,q} \cap E_{n,q})} \int_{h_{n,q}(\Delta_{i,q} \cap E_{n,q})} \varphi \, d\lambda \right.$$
$$\left. - \sum_{\Delta_{i,q} \subset J^{(x)}_{n,q}} \frac{1}{q^d} \frac{1}{\lambda(\Delta_{i,q} \cap E_{n,q})} \int_{h_{n,q}(\Delta_{i,q} \cap E_{n,q})} \varphi \, d\lambda \right| < \frac{q^d}{q'} \|\varphi\|_0$$

$$\left| \frac{1}{q'} \sum_{i=0}^{q'-1} \varphi(h_{n,q} S^i_{1/q'} x) - \frac{1}{q'} \sum_{i \in \mathcal{O}^{(x)}} \varphi(h_{n,q} S^i_{1/q'} x) \right| < \frac{q^d}{q'} \|\varphi\|_0$$

Second we produce estimates using the fact that $\mathcal{O}^{(x)}$ is equidistributed among the elements of η_q and using (23) and (24)

$$\left| \frac{1}{q'} \sum_{i \in \mathcal{O}^{(x)}} \varphi(h_{n,q} S^i_{1/q'} x) - \frac{1}{q'} \sum_{i \in I} \varphi(h_{n,q} S^i_{1/q'} x) \right| < \frac{4d}{n^2} \|\varphi\|_0,$$

$$\left| \int_{h_{n,q} J^{(x)}_{n,q}} \varphi \, d\lambda - \int \varphi \, d\lambda \right| < \frac{4d}{n^2} \|\varphi\|_0$$

Finally we produce estimates using (25)

$$\left|\int_{h_{n,q}(J_{n,q}^{(x)}\cap E_{n,q})}\varphi\,d\lambda-\int_{h_{n,q}J_{n,q}^{(x)}}\varphi\,d\lambda\right|<\frac{2(d-1)}{n^2}\|\varphi\|_0,$$

$$\left|\frac{1}{q^d\lambda(\Delta_{i,q}\cap E_{n,q})}\int_{h_{n,q}(J_{n,q}^{(x)}\cap E_{n,q})}\varphi\,d\lambda-\int_{h_{n,q}(J_{n,q}^{(x)}\cap E_{n,q})}\right|<\frac{4(d-1)}{n^2}\|\varphi\|_0.$$

Combining these estimates gives us exactly (27) as required. □

Proof of Theorem 2: Let $\Phi=\{\varphi_n\}$ be a set of Lipshitz functions that is dense in $C^0(\mathbb{T}^d,\mathbb{R})$. Let L_n be a uniform Lipshitz constant for $\varphi_1 H_n,\dots,\varphi_n H_n$.

At step n we can choose q_n' so that

1. $q_n^d>n^2 L_n\sqrt{d}$,
2. $q_n'>n^2 q_{n-1}^d$.

Both of these depend solely on the size of q_n' which we are free to make arbitrarily large and so these conditions are compatible with the proof of Proposition 2. The first of our two size conditions implies that $\varphi H_n,\dots,\varphi_n H_n$ are uniformly $(\sqrt{d}q_n^{-d},n^{-2})$-continuous. Therefore we can apply Proposition 3 with $q=q_n$ and $q'=q_{n+1}$ and $\epsilon=n^{-2}$ to conclude that for $1\le k\le n$ and for all $x\in\mathbb{T}^d$

$$\left|\frac{1}{q_{n+1}'}\sum_{i=0}^{q_n'-1}\varphi_k H_n(h_{n,q_n}S^i_{1/q_{n+1}'}x)-\int\varphi_k H_n d\lambda\right|<\frac{14d}{n^2}\|\varphi_k\|_0+\frac{2q_n^d}{q_{n+1}'}\|\varphi_k\|_0+\frac{2}{n^2}.$$

Using the fact that H_n is measure-preserving, replacing x by $H_{n+1}x$, and reordering the orbit we obtain for $1\le k\le n$ and for all $x\in\mathbb{T}^d$

$$\left|\frac{1}{q_{n+1}'}\sum_{i=0}^{q_{n+1}'-1}\varphi_k(T_{n+1}^i x)-\int\varphi_k d\lambda\right|<\frac{16d}{n^2}\|\varphi_k\|_0+\frac{2}{n^2}.$$

This establishes (21) from Lemma 5. To establish (22) from 5 observe that there exist $C>0$ and $m\in\mathbb{N}$ such that

$$\begin{aligned}d^{(q_n')}(T_n,T_{n+1})&\le |||H_{n+1}|||_1 q_n'|\alpha_n-\alpha_{n+1}|\\&\le C\,q_n^m\,|\alpha_n-\alpha|\end{aligned}$$

and hence we can choose q_n' so that $d^{(q_n')}(T_n,T_{n+1})<1/n^2$. By the triangle inequality we immediately obtain that $d^{(q_n')}(T_n,T)<2/n$. In actual fact this estimate is weaker than those that arise in the proof of Proposition 2 and so is automatic.

This verifies the hypotheses of Lemma 5 and hence we conclude that T is uniquely ergodic. □

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