MINIMAL BUT NOT UNIQUELY ERGODIC DIFFEOMORPHISMS

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Abstract. This paper provides a method of constructing smooth minimal diffeomorphisms whose set of ergodic measures has a given cardinality. The construction applies to compact manifolds admitting free S^1-actions. The diffeomorphisms constructed all lie in the closure of the space of periodic diffeomorphisms. However they cannot be obtained by Baire Category arguments.

1. Introduction

If f is a uniquely ergodic transformation on a separable metric space preserving a Borel measure μ then $f|_{\mathrm{supp}(\mu)}$ is minimal, that is every orbit is dense [7, Proposition 4.1.18]. It was therefore a natural question whether minimality is sufficient for unique ergodicity. This is one in a series of questions which ask which topological properties are sufficient for more abstract metric properties. The unfortunate answer has been that while metric properties have strong topological consequences the converse is false. One reason for this is that there exist smooth cocycles which are measurable coboundaries but for which the transfer functions behave wildly topologically. This was known by A. N. Kolmogorov [1] who used this method to construct a time change of a linear flow on $\mathbb{T}^2$ with pure point spectrum but with discontinuous eigenfunctions. The same observation was independently made by H. Furstenberg [3] who used it to construct diffeomorphisms which are minimal but not uniquely ergodic (see [7, Corollary 12.6.4] for the essence of the construction). These diffeomorphisms are skew-products, are analytic, and all admit uncountably many ergodic measures.

A construction of minimal transformations with any given finite number, a countable number, or a continuum of ergodic invariant measures was provided by S. Williams [6] in the case of symbolic dynamics. This serves as very general counterexample to our earlier question in the case of symbolic systems. No such construction has been published for the smooth category. A construction of topologically transitive diffeomorphisms preserving a smooth measure and with a given number of ergodic components was announced by D. V. Anosov and A. B. Katok [4] without proof. We use a variation of their methods to construct minimal diffeomorphisms with a given number of ergodic measures. The idea of transversal cutting which is one of the keys for avoiding the measure zero exceptional set which occurs in [4] was communicated to the author by A.B Katok in lectures on smooth ergodic theory given at The Pennsylvania State University during 1998.

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2. The Theorem

Theorem. *Let $\alpha \in \mathbb{Q}$, M be a compact and connected smooth manifold admitting a free C^∞ action of S^1, $\tau : S^1 \times M \to M$, and μ be a smooth positive measure invariant under τ. Let $d \in \mathbb{N}$*

Then there exists a diffeomorphism T, arbitrarily close to $\tau_\alpha(\cdot) := \tau(\alpha, \cdot)$ which is minimal but has exactly d ergodic invariant measures which are absolutely continuous with respect to μ.

3. Structure of the Construction

The required diffeomorphism is constructed using a variant of the fast approximation - conjugation method pioneered by Anosov and Katok [4]. The required diffeomorphism T is obtained as the limit of a sequence of periodic diffeomorphisms, $\{T_n\}$, each of which preserve μ. Each periodic diffeomorphism T_n is smoothly conjugate to τ_{α_n} for some $\alpha_n \in \mathbb{Q}$,

$$T_n = h_n \circ \tau_{\alpha_n} \circ h_n^{-1}.$$

Let

$$\mathfrak{B}(M) := \{h \circ \tau_\alpha \circ h^{-1} : \alpha \in S^1, h \in \mathrm{Diff}^\infty(M)\} \subset \mathrm{Diff}^\infty(M).$$

By construction $T \in \overline{\mathfrak{B}(M)}$. Fathi and Herman showed in [5] that the collection of uniquely ergodic maps forms a dense G_δ in $\overline{\mathfrak{B}(M)}$. Thus unique ergodicity is generic in $\mathfrak{B}(M)$. This remains true if we restrict ourselves to $\mathfrak{B}(M, \mu) \subset \mathfrak{B}(M)$ where the conjugacy h is taken to preserve μ. Thus our construction is not generic.

The sequence of conjugacies, $\{h_n\}$, and “rotation numbers”, $\{\alpha_n\}$ are constructed inductively.

$$h_n = A_1 \circ \cdots \circ A_n$$

where each A_i preserves μ. The “rotation numbers” satisfy

$$\alpha_{n+1} = \alpha_n + \frac{1}{m_n Q_n q_n}$$

where q_n is the denominator of α_n and m_n and Q_n are parameters defined later.

We take a partition of our space into d Borel sets of positive measure. These naturally support d absolutely continuous probability measures $\{\mu_i\}$, given by the normalized restrictions of μ. The required d ergodic invariant measures are constructed from the μ_i as the limits of the sequences

$$\xi_i^n := (h_n)_* \mu_i.$$

Notice that ξ_i^n is preserved by T_n.

4. Topological Preliminaries

The approach of Anosov and Katok applies to trivial bundles. For constructions in the measurable category this is not really a restriction since every smooth bundle is trivial in the measurable category. Furthermore any manifold admitting a smooth periodic flow is measurably a trivial bundle. Unfortunately our construction is not a measurable category construction and we must perforce deal with non-trivial bundles. These topological lemmas are part of our extension of the Anosov and Katok approach to the case of non-trivial bundles.

The approach of Anosov and Katok allowed for manifolds with boundary. Unfortunately our approach cannot deal with these. Obviously there cannot exist minimal diffeomorphisms of the disk since the boundary must be preserved and there is always a fixed point. However it might be supposed that the orbit of any other point could be made dense. That this is not possible was shown by Le Calvez and Yoccoz [8]. The ergodic properties of such a diffeomorphism may be more tractable.

Lemma 1. *If N is a connected compact Riemannian manifold then there exist a constant K such that for all $\epsilon > 0$ there exists K embedded disks $\{D_i\}$ such that*

(1) $\operatorname{vol}(D_i) > 1 - \epsilon$ *where* vol *is the normalized Riemannian volume on N.*
(2) $\{D_i^\circ\}$ *is an open cover of N.*

Proof. As N is a smooth compact Riemannian manifold there exists a smooth triangulation of N [2]. Let $\{\mathcal{S}_i\}_{i=0}^m$ be the set of simplices from the triangulation with the same dimension as N. For each simplex $\mathcal{S}_i$ construct a smoothly embedded closed disk K_i by taking a slightly smaller simplex within $\mathcal{S}_i$ and smoothing the corners. K_i can be constructed such that $\operatorname{vol}(K_i) > (1-\epsilon)\operatorname{vol}(\mathcal{S}_i)$.

Construct a graph with vertices $\{\mathcal{S}_i\}$ by putting an edge between $\mathcal{S}_i$ and $\mathcal{S}_j$ if they share a face. This is a connected graph which we will call the adjacency graph. Construct a maximal tree in the adjacency graph. We will call this the joining tree.

We describe the step by step procedure for constructing the disk D_j. Each disk D_j is constructed by joining the disks K_i described above but enlarging one disk so that the family forms a cover.

(1) Using coordinate charts construct a smoothly embedded disk κ_j such that $K_i \cap \kappa_j = \varnothing$ for all $i \neq j$ and κ_j° is a neighborhood of $\mathcal{S}_j$.
(2) For each edge in the joining tree not involving $\mathcal{S}_j$ construct a smooth tube which joins the appropriate K_i and K_k. Let U be a chart neighborhood about the center of their common face. Using the smooth coordinates in the interior of each simplex construct a smooth tube joining the appropriate K to the neighborhood U. Use the smooth coordinates in U to complete the tube.
(3) For each edge in the joining tree joining a K_i to κ_j we can construct a smooth tube using the smooth coordinates in the interior of $\mathcal{S}_i$.

This yields a smoothly embedded disk since we can show that the addition of a tube and disk to an embedded disk constructed by joining K_i and K_k is again an embedded disk and then proceed by induction. Let us imagine joining $K_i \subset \mathcal{S}_i$ to a smooth embedded disk via a tube which is contained in $K_i \cup K_k$ using the process outlined in step 2 above. Using the smooth coordinates in the interior of $\mathcal{S}_k$ we can show that the combination of the tube and K_k is diffeomorphic to a short tube with a spherical cap contained within the neighborhood U and this diffeomorphism extends via the identity. Now use the smooth coordinates in U to contract the tube and cap to a tube and cap contained in the interior of $\mathcal{S}_i$. Finally we use the smooth coordinates in the interior of $\mathcal{S}_i$ to show that the smooth embedded disk plus a tube and cap is diffeomorphic to the original embedded disk. This argument covers the joining of K_i and K_j but we can show using coordinate charts that connecting K_i and κ_j is equivalent to connecting K_i and K_j whereupon the earlier argument applies.

Thus we get a smoothly embedded disk with

$$\operatorname{vol}(D_j) > \sum \operatorname{vol}(K_i) > (1-\epsilon)\sum \operatorname{vol}(\mathcal{S}_i) = 1 - \epsilon.$$

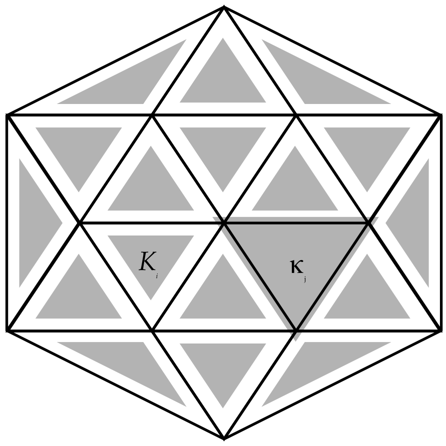

FIGURE 1. An example of the a triangulation with disks $\{K_i\}$ and disk κ_j.

Since D_i° contains a neighborhood of the simplex $\mathcal{S}_i$ the family $\{D_i\}$ covers N. The constant K is precisely the number of highest dimensional simplices in the triangulation of N. $\square$

Let $B(r)$ denote the standard closed disk of radius r in $\mathbb{R}^{\dim N}$.

Lemma 2 (Legerdemain Lemma). *Let N be a smooth connected compact manifold without boundary endowed with a smooth positive measure μ. Given any positive integer d there exists a cover of N by d sets $\{N_i\}$ such that*

(1) *each set N_i is closed and connected.*
(2) $N_i^\circ \cap N_j = \varnothing$ *for all* $i \neq j$.
(3) N_i *is a smoothly embedded disk for* $1 \leq i \leq d-1$. *In particular* ∂N_i *is an embedded sphere.*
(4) *there exists K such that for all $\epsilon > 0$ there is a cover of N by K smoothly embedded closed disks $\{D_i\}$, such that $\mu(D_i) > 1 - \epsilon$ and $N_i \subset D_j^\circ$ for $1 \leq i \leq d-1$ and for all j.*

Proof. Construct a triangulation, adjacency graph, and joining tree as in Lemma 1. Let $\{\mathcal{S}_i\}$ be the set of simplices from the triangulation with the same dimension as N. Fix $\epsilon_1 > 0$ and consider the collection of smooth embedded closed disks L_i constructed by taking a smaller simplex in $\mathcal{S}_i$ and smoothing the corners such that $\mu(L_i) > (1-\epsilon_1)\mu(\mathcal{S}_i)$. Now in any cover constructed using Lemma 1 (using μ in place of vol) with $\epsilon > \epsilon_1$ we will have $L_i \subset K_i^\circ$. Let $\{d_j\}$ be the family of embeddings, $d_j : B(1) \to N$, given by Lemma 1. There are only finitely many of these so, by continuity, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_O$ and each j the embedded disk $d_j\big(B(1-\delta)\big)$ is connected and contains $\cup L_i$. Taking any $d-1$ disjoint smoothly embedded disks N_i in $\cup L_i$ and adding $N_d = N \setminus \bigcup_{i=1}^{d-1} N_i^\circ$ we obtain a cover with the required property. $\square$

5. Preliminary Construction on a Trivial Sub-Bundle

The Legerdemain lemma reduces each step of the construction to several simpler sub-steps. Each sub-step is free of topological obstructions. Each step in our iterative construction is built up of a finite number of sub-steps each of which consists of a construction on a trivial sub-bundle.

Let $B(1)\times S^1$ be a trivial bundle endowed with normalized Lebesgue measure, λ. Let $\{R_1,\dots,R_{n-1}\}$ be a collection of disjoint smoothly embedded closed disks contained in $B(1)^\circ$ and let $R_n = \overline{B(1)\setminus\bigcup_{i=1}^{n-1} R_i}$. Define $\tilde{C} := B(1)\times S^1$ and $\tilde{C}_j := R_j\times S^1$. Each $\tilde{C}_j$ supports a measure given by

$$\nu_j(A) := \frac{\lambda(\tilde{C}_j\cap A)}{\lambda(\tilde{C}_j)}.$$

Lemma 3 (Sub-Bundle Lemma)**.** *Let $\alpha=\frac{p}{q}\in\mathbb{Q}^+$, with $(p,q)=1$, be given, and $\Phi=\{\varphi_1,\dots,\varphi_n\}$ be a given finite set of continuous functions on $\tilde{C}$.*

Then for all $\epsilon>0$ there exists $\delta_0>0$ such that for all $0<\delta<\delta_0$ there exists $Q\in\mathbb{N}$, and a measure-preserving diffeomorphism A such that

(1) *there exists a τ_α-invariant set Σ with $B(1-\delta)\times S^1\subset\Sigma\subset B(1-\frac{\delta}{2})\times S^1$ such that A is the identity off Σ.*
(2) $A\circ\tau_\alpha=\tau_\alpha\circ A$.
(3) $\forall\varphi\in\Phi$, $\forall x\in\Sigma$, *and* $\forall\alpha'=\frac{p'}{q'}$ *with* $q'=mQq$ *and* $(p',q')=1$

$$\inf_{\nu\in\mathfrak{M}}\left|\frac{1}{q'}\sum_{\Pi^x}\varphi\big(A\circ\tau^i_{\alpha'}x\big)-\frac{\mathrm{Card}(\Pi^x)}{q'}\int\varphi\,d\nu\right|<\epsilon$$

where $\mathfrak{M}$ is the simplex of measures generated by $\{\nu_1,\dots,\nu_d\}$, $m\in\mathbb{N}$ is arbitrary, and $\Pi^x:=\{i:\tau^i_{\alpha'}x\in\Sigma\}$.
(4) $\forall\alpha'=\frac{p'}{q'}$, *with $q'=mQq$ and $(p',q')=1$, the map $A\circ\tau_{\alpha'}\circ A^{-1}$ is ϵ-minimal, that is every orbit intersects every ball of radius ϵ.*
(5) $\forall\varphi\in\Phi$ *and* $1\le i\le d$

$$\left|\int\varphi\,d\nu_i-\int\varphi\circ A\,d\nu_i\right|<\epsilon.$$

(6) $\lambda(A\tilde{C}_i\Delta\tilde{C}_i)<\epsilon$ *for* $1\le i\le d$.

5.1. **Outline of the Proof of the Sub-Bundle Lemma.** This lemma is the key step in proving Theorem 1. It is proved by constructing appropriate combinatorics on the bundle and then constructing a smooth realization of these combinatorics.

5.1.1. *Combinatorics.* In order to define combinatorics on the bundle it must be discretized. Cover the disk $B(1-\delta)$ by closed cubes of side length s, $\{C_i\}$, such that $C_i^\circ\cap C_j=\varnothing$ and $C_i^\circ\cap B(1-\delta)\neq\varnothing$. Choose s small enough so that $B(1-\delta)\subset\cup C_i\subset B(1-\delta/2)$. Later there with be further restrictions on s.

Associate each cube C_i with a region R_j. A cube C_i can be associated with a region R_j provided $C_i^\circ\cap R_j\neq\varnothing$. Let the sets of cubes associated to each R_j be connected. Let N_i denote the number of cubes associated with R_i, $N=\sum N_i$, and $L=\mathrm{lcm}(N_1,\dots N_n)$. Renumber the cubes C_i so that the first N_1 are associated with R_1, the next N_2 with R_2, and so on.

This produces a partial partition of $\Delta_{lq} = B(1) \times [0, \frac{1}{lq})$ into cubes, $Q_{i,j}$ where $1 \le i \le N,\ 1 \le j \le N + kL$, by

$$Q_{i,j} = \{(x,t) : x \in C_i, \frac{j-1}{ql(N+kL)} \le t < \frac{j}{ql(N+kL)}\}.$$

This can be envisioned as a tower with the base $\cup C_i \subset B(1)$ and $N + kL$ floors of equal height.

Define A on Δ_{lq} and extend it to all of $\tilde{C}$ by requiring

$$A \circ \tau_\alpha = \tau_\alpha \circ A.$$

Outside of the union of the cubes define the map to be the identity. On the union of the cubes, $\cup Q_{i,j} \subset \Delta_{lq}$, choose A so that it induces the following permutation

$$A(Q_{i,j}) = \begin{cases} Q_{\alpha^{j-1}(i),j} & j \le N \\ Q_{\beta^{j-N-1}(i),j} & \text{otherwise} \end{cases} \tag{1}$$

where

$$\begin{aligned} \alpha &= (1\ \dots\ N) \\ \beta &= (1\ \dots\ N_1)(N_1+1\ \dots\ N_1+N_2)\ \dots\ (N-N_n+1\ \dots\ N_n) \end{aligned}$$

are permutations in the symmetric group on N elements. There is no restriction on the action of A within the cubes. Further properties, such as weak mixing, would require permuting the cubes between the levels of Δ_{lq}.

The cycle α is applied on the first N levels of $N+kL$ total levels in Δ_{lq}. Though the entire cycle is traversed only once in Δ_{lq}, there are lq such regions in $\tilde{C}$. Hence the frequency of these regions is determined by l. This region will give the ϵ-minimality. Since the levels all have equal measure, the measure of the area on which α is applied goes to 0 as $k \to \infty$. The idea behind the construction is to make this region occur frequently but have small measure, thus making this region topologically significant but measure theoretically insignificant.

On the remaining levels in the tower the permutation β is applied. This cyclically permutes the cubes corresponding to each region, R_j. This ensures that on some scale the orbits are uniformly distributed on $\tilde{C}_j$ with the small measure exception corresponding to the region on which α is applied. This is the key in proving conclusion 3 of Lemma 3 and is embodied in the main estimate (7).

5.1.2. *Smooth Realization.* As intuitively appealing as the above construction is, it is clearly not smooth. To smooth the construction it is necessary to relinquish control over part of the space. The cubes $Q_{i,j}$ are replaced by smaller cores which can be permuted as above by a C^∞-diffeomorphism of M. In order to retain sufficient control to be able to produce conclusion 3 it is necessary to control a large proportion of *every* orbit. This is where the construction leaves the measurable category since for such a construction it would suffice to control most of the orbits at each step. This leads to control of *almost all* orbits of the limiting transformation. A measurable construction would be able to neglect a measure zero set of orbits. However for our construction this is fatal. A single orbit could support an ergodic invariant measure. A construction in the measurable category is sufficient to control the number of *absolutely continuous* ergodic invariant measures.

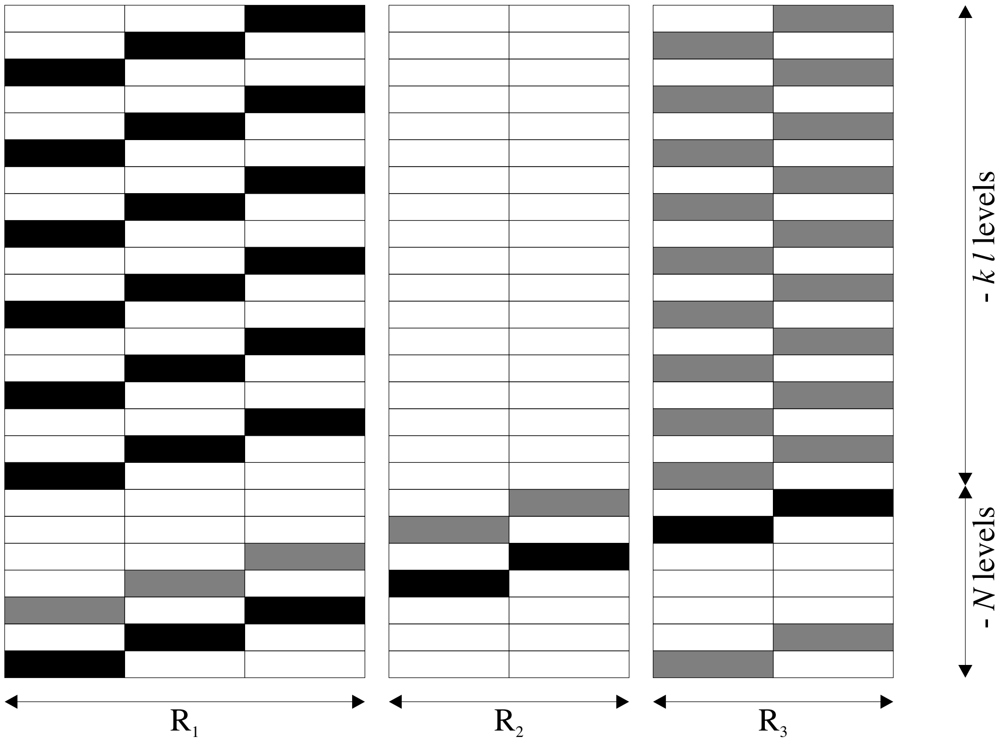

FIGURE 2. An example of the action of A where $d = 3$, $N_1 = 3$, $N_2 = N_3 = 2$, and $k = 3$.

Let s be the side length of our decomposition and v be a diagonal vector of one of our cubes of side length s. Take a collection of cubes of side length s, $\{C_i\}$, such that $\cup_{t\in[0,1/2)}(B(1-\delta)-tv)\cap C_i \neq \emptyset$ and $\cup_{t\in[0,1/2)}(B(1-\delta)-tv) \subset \cup_i C_i \subset B(1-\delta/2)$.

As before, A is defined on Δ_{lq} and extended to $\tilde{C}$ by requiring it commute with $\tau_{\frac{1}{lq}}$. Partially partition Δ_{lq} by parallelepipeds

$$P_{i,j} := \Big\{(x,t) : x - \frac{lq(N+kL)t+1-j}{2}v \in C_i,\ \frac{j-1}{lq(N+kL)} \leq t < \frac{j}{lq(N+kL)}\Big\}.$$

Scale each parallelepiped about its center by a factor γ. This gives a family of identical disjoint parallelepipeds $\{\tilde{Q}_{i,j}\}$. These are called cores and play the role of the $Q_{i,j}$ in the previous section.

Define the map A to be a diffeomorphism inducing the same permutations as before (1). Any two adjacent parallelepipeds can be enclosed in a region diffeomorphic to a disk such that the images of the two parallelepipeds are related via a rotation about the center. Since any permutation of cores can be written as a product of such transpositions A can be taken to be a measure preserving C^∞-diffeomorphism. Defined in the this manner the diffeomorphism A automatically satisfies conclusions 1 and 2 of Lemma 3.

5.2. **Proof of the Sub-Bundle Lemma.** There are several parameters of the construction: δ_0, s, γ, k, P, and l. The parameters δ_0, s, P, and γ are related to

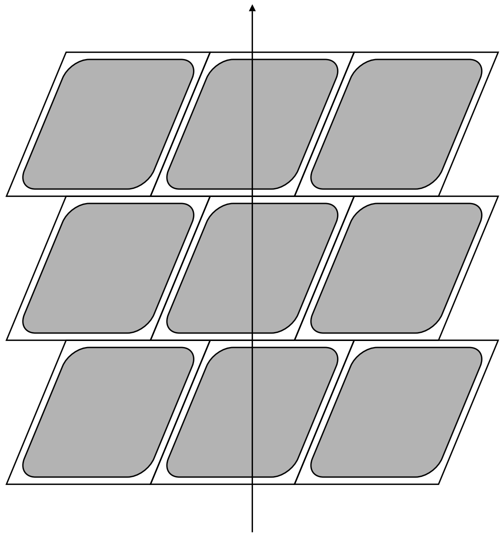

FIGURE 3. Example of cores. Arrow denotes the fibre direction. Notice the gaps are transverse to the fibre direction.

the geometry. The parameters k and l govern the combinatorics. Let $\kappa \geq 1$ be a global bound for $|\varphi|$ for all $\varphi \in \Phi$.

5.2.1. *Geometric Estimates.* Fix δ_0 so that for $1 \leq j \leq n$

$$\nu_j\big(B(1-\delta_0)\big) < \frac{\epsilon}{20\kappa}$$

and for $1 \leq i \leq n-1$ we have

$$R_i \subset B(1-\delta_0).$$

As the boundary of R_j is a smooth compact sub-manifold we can choose s sufficiently small such that the union of the parallelepipeds associated to $\tilde{C}_j$, denoted $\tilde{V}_j$, approximates $\tilde{C}_j$ in the sense that

$$\frac{\lambda(\tilde{C}_j \Delta \tilde{V}_j)}{\lambda(\tilde{C}_j)} < \frac{\epsilon}{18\kappa} \tag{2}$$

Consideration of the action of A leads naturally to the following partial partition of Δ_{lq} into "columns",

$$K_{a,b} := \begin{cases} \bigcup_{m=1}^{N} P_{a,m} & b = 1 \\ \bigcup_{m=1}^{L} P_{a,N+(b-2)L+m} & b \geq 2 \end{cases}$$

This is extended to a partial partition of $B(1) \times S^1$ by requiring it to commute with $\tau_{\frac{1}{lq}}$. Set

$$K_{a,b,c} := \tau_{\frac{1}{lq}}^{c} K_{a,b}.$$

The columns $K_{a,b,c}$ with $b = 1$ correspond to levels on which the cycle α-cycle is applied. The orbit of $A \circ \tau_\alpha \circ A^{-1}$ is almost equidistributed among the $\{K_{a,b,c} \subset \tilde{V}_i : b \neq 1\}$. This observation together with the following local estimate (3) are the key to proving the main estimate (7).

From the geometry of the parallelepipeds there exists a constant $C > 0$ which depends solely on the dimension of M such that $\operatorname{diam}(K_{a,b,c}) < Cs^{\dim M}$. Uniform continuity guarantees that $\operatorname{var}_{K_{a,b,c}} \varphi < \epsilon/9$ for all $\varphi \in \Phi$ given a sufficiently small s. Thus for all $\varphi \in \Phi$ and $y \in K_{a,b,c} \subset \tilde{V}_j$ the following local estimate holds

$$\Big|\frac{1}{\nu_j(K_{a,b,c})}\int_{K_{a,b,c}} \varphi \, d\nu_j - \varphi(y)\Big| < \frac{\epsilon}{9}. \tag{3}$$

Choosing P, the minimum number of points of $\tau_{\alpha'}$ within a level of our tower (corresponding to $m = 1$), to be large enough we can make the proportion of the orbit contained within cores arbitrarily close to the proportion of the fiber contained within the cores. This can be made arbitrarily close to 1 by choosing γ sufficiently close to 1.

Let $Q := l(N + kL)P$.

5.2.2. *Combinatorial Estimates.* Let

$$\Gamma := \bigcup_{i=0}^{lq-1} \tau^i_{\frac{1}{lq}}\Big(B(1) \times \Big[\frac{N}{lq(N+kL)}, \frac{1}{lq}\Big)\Big).$$

Choose k sufficiently large that the proportion of a τ-orbit which lies in the levels of the tower which correspond to the cycle α satisfies

$$\lambda(\tilde{C}\backslash\Gamma) = \frac{N}{N+kL} < \frac{\epsilon}{9\kappa}. \tag{4}$$

Given the orbit of x under $\tau_{\alpha'}$ denote by π_1^x the collection of points on the orbit which lie in levels on which the cycle α is applied, π_2^x the collection of points which are not contained in the cores, and π_3^x the collection of points in cores in levels which correspond to the permutation β. Let $\Pi_j^x = \{i : A \circ \tau^i_{\alpha'}(x) \in \pi_j^x\}$. By choosing γ sufficiently close to 1, P large enough, and k large enough we can guarantee for all l that

$$\frac{\operatorname{Card}(\Pi_1^x \cup \Pi_2^x)}{q'} < \frac{\epsilon}{3\kappa} \qquad\qquad \frac{\operatorname{Card}(\Pi_3^x)}{q'} \geq \frac{\operatorname{Card}(\Pi^x)}{q'} - \frac{\epsilon}{3\kappa}. \tag{5}$$

Let the proportion of π_3^x that lies within $\tilde{V}_j$ be ω_j. The proportion of π_3^x contained in a column $K_{a,b,c} \subset \tilde{V}_j$ is, for $b \neq 1$

$$\rho_j := \frac{L}{lq(N+kL)}\frac{\omega_j}{N_j}.$$

Taking the sum over $\Pi_1^x \cup \Pi_2^x$ yields

$$\Big|\sum_{i\in\Pi_1^x\cup\Pi_2^x} \varphi(A \circ \tau^i_{\alpha'}x)\Big| < \kappa \ \operatorname{Card}(\Pi_1^x \cup \Pi_2^x). \tag{6}$$

Taking a similar sum over Π_3^x and using the local estimate (3) yields

$$\Big|\sum_{\Pi_3^x} \varphi(A \circ \tau^i_{\alpha'}x) - \operatorname{Card}(\Pi_3^x)\sum_{j=1}^{d} \frac{\rho_j}{\nu_j(K)}\int_{\tilde{V}_j\cap\Gamma} \varphi \, d\nu_j\Big| < \frac{\epsilon}{9} \ \operatorname{Card}(\Pi_3^x). \tag{7}$$

The geometry of the construction means that the measure of a column on which the permutation β is applied, $K = K_{a,b,c}$ with $b \neq 1$, is given by

$$\nu_j(K) = \frac{\nu_j(\tilde{V}_j)L}{lq(N+kL)N_j}$$

where $\tilde{V}_j$ is the "cylinder" containing the column.

This reduces the main estimate (7) to

$$\Big|\sum_{i\in\Pi_3^x}\varphi(A\circ\tau_{\alpha'}^i x) - \mathrm{Card}(\Pi_3^x)\sum_{i=1}^{s}\frac{\omega_i}{\nu_i(\tilde{V}_i)}\int_{\tilde{V}_i\cap\Gamma}\varphi\,d\nu_i\Big| < \frac{\epsilon}{9}\ \mathrm{Card}(\Pi_3^x). \tag{8}$$

Use (2) and (4) to get

$$\Big|\frac{1}{\nu_i(\tilde{V}_i)}\int_{\tilde{V}_i\cap\Gamma}\varphi d\nu_i - \int\varphi d\nu_i\Big| < \frac{2\epsilon}{9}.$$

Define $\nu := \sum_{i=1}^n \omega_i\nu_i$ and apply (8) to get

$$\Big|\sum_{i\in\Pi_3^x}\varphi(A\circ\tau_{\alpha'}^i x) - \mathrm{Card}(\Pi_3^x)\int\varphi d\nu\Big| < \frac{\epsilon}{3}\ \mathrm{Card}(\Pi_3^x). \tag{9}$$

The cardinality estimate for Π_3^x , (5), implies that

$$\Big|\Big(\frac{\mathrm{Card}(\Pi^x)}{q'} - \frac{\mathrm{Card}(\Pi_3^x)}{q'}\Big)\int\varphi\,d\nu\Big| < \frac{\epsilon}{3}.$$

Taking the estimates for the sums over $\Pi_1^x\cup\Pi_2^x$, (6), and Π_3^x, (9), dividing by q', and applying the previous estimate yields the desired property

$$\Big|\frac{1}{q'}\sum_{i\in\Pi^x}\varphi(A\circ\tau_{\alpha'}^i x) - \frac{\mathrm{Card}(\Pi^x)}{q'}\int\varphi\,d\nu\Big| < \epsilon.$$

To prove ϵ-minimality it is convenient to use the metric on $\tilde{C}$ given by

$$d((x,t),(x',t')) := \max\{d(x,x'),|t-t'|\}.$$

In this metric, which is equivalent to the usual metric, the ϵ-ball is a cylinder. Choosing $l > 2/\epsilon$ guarantees that the cylinder passes through a collection of levels on which the entire α-cycle is traversed. If s is chosen small enough then the cylinder must contain an entire column from this collection. Since every orbit meets every column, every orbit meets every ϵ-ball.

Observe that A preserves the decomposition into $\{\tilde{V}_i\}$ except on the levels corresponding to the cycle α. From (4) it follows that

$$\frac{\lambda(\tilde{V}_i\Delta A\tilde{V}_i)}{\lambda(\tilde{C}_i)} < \frac{2\epsilon}{9\kappa}.$$

By the triangle inequality and (2) we have

$$\frac{\lambda(\tilde{C}_i\Delta A\tilde{C}_i)}{\lambda(\tilde{C}_i)} < \frac{1\epsilon}{3\kappa}.$$

This implies conclusions 5 and 6.

This completes the proof of the lemma. □

6. Extension to the Entire Bundle

In this section we state and prove a proposition which extends the lemma above to the entire bundle. The proof is an induction over the disks provided by Lemma 2 applying Lemma 3 at each step.

Choose a cover of N by a d sets, $\{N_i\}$, using Lemma 2.

Proposition 4. *Let $\alpha = \frac{p}{q} \in \mathbb{Q}^+$, with $(p,q) = 1$, be given, and $\Phi = \{\varphi_1, \dots, \varphi_n\}$ be a given finite set of continuous functions on M.*

Then for all $\epsilon > 0$ there exists $Q \in \mathbb{N}$, and a diffeomorphism A of M such that

(1) *$\forall \alpha' = \frac{p'}{q'}$, with $q' = mQq$ and $(p', q') = 1$, the map $A \circ \tau_{\alpha'} \circ A^{-1}$ is ϵ-minimal, that is every orbit intersects every ball of radius ϵ.*

(2) *$\forall \varphi \in \Phi$, $\forall x \in M$, and $\forall \alpha' = \frac{p'}{q'}$ with $q' = mQq$ and $(p', q') = 1$*

$$\inf_{u \in \mathfrak{M}'} \Big| \frac{1}{q'} \sum_{i=0}^{q'-1} \varphi\big(A \circ \tau_{\alpha'}^i x\big) - \int \varphi \, du \Big| < \epsilon$$

where $\mathfrak{M}'$ is the simplex of measures generated by $\{\mu_1, \dots, \mu_n\}$.

(3) *$A \circ \tau_\alpha = \tau_\alpha \circ A$.*

(4) *$\mu(A\tilde{N}_i \Delta \tilde{N}_i) < \epsilon$ for $1 \le i \le d$.*

6.1. **Preliminaries.** Construct D_j using Lemma 2 so that for all i, j

$$1 - \tilde{\mu}_i(D_j) < \min\Big\{\frac{\epsilon}{6K\kappa}, \mathrm{vol}\big(\frac{\epsilon}{2}\big)\Big\} \tag{10}$$

where $\mathrm{vol}\big(\frac{\epsilon}{2}\big)$ is the least measure of a $\frac{\epsilon}{2}$-ball in M and K is the number of disks required for N. Consider the normalized pull-back of the measure $\tilde{\mu}$ via the embedding d_i. This gives a smooth positive measure on $B(1)$ which, by Moser's Theorem [7, Theorem 5.1.27], is the push-forward of the Lebesgue measure on $B(1)$ by a diffeomorphism m_i. Let $\rho_i = d_i \circ m_i$ be the new measure-preserving embedding. Consider its extension to a map $\tilde{\rho}_i : \tilde{C} \to M$ defined by $\tilde{\rho}_i(x,t) = \tau_t \rho_i(x)$. By uniform continuity of $\tilde{\rho}_i$ there exists an ϵ' such that the pull-back of any $\epsilon/2$-ball centered at $x \in \tilde{D}_i$ contains a ball of radius ϵ'.

Let ζ_i^j denote the push-forward of ν_i on via $\tilde{\rho}_j$. The measures $\zeta_1^j, \dots, \zeta_d^j$ on the cylinder $\tilde{D}_j = \pi^{-1}(D_j) \subset M$ are related to the measures $\mu_i, \dots, \mu_d$ by

$$\zeta_i^j(A) = \frac{\mu_i(\tilde{D}_j \cap A)}{\mu_i(\tilde{D}_j)}$$

From (10) it is immediate that

$$\Big| \int_{\tilde{D}_j} \varphi \, d\zeta_i^j - \int_{\tilde{D}_j} \varphi \, d\mu_i \Big| < \frac{\epsilon}{6K} \qquad \Big| \int_{\tilde{D}_j} \varphi \, d\mu_i - \int_M \varphi \, d\mu_i \Big| < \frac{\epsilon}{6K}$$

and thus we have for each $\tilde{D}_j$

$$\Big| \int_{\tilde{D}_j} \varphi \, d\zeta_i - \int_M \varphi \, d\mu_i \Big| < \frac{\epsilon}{3K} \tag{11}$$

Henceforth Lemma 3 will be treated as applying to the $\tilde{D}_j$ and giving diffeomorphisms on M. The diffeomorphisms given by Lemma 2 are identity on a neighborhood of the boundary and hence can be extended by the identity to diffeomorphisms on M. Since $\{D_j^\circ\}$ is an open cover of N there exists $\delta_1 > 0$ so that $\{D_j' := d_j(B(1-\delta_1))\}$ still cover N. For each application of Lemma 3 we will take $\delta < \min(\delta_0, \delta_1)$. We adopt our earlier convention and denote the associated trivial bundles by $\tilde{D}_j'$.

6.2. **Details.** Denote the α given in the Proposition by α_0.

6.2.1. *Induction Hypotheses.* Assume there are diffeomorphisms $A_1, \dots, A_j$ with their associated sets $\Sigma_1, \dots, \Sigma_j$ from Lemma 3 such that

(1) for all $x \in \bigcup_{i=1}^{j} \tilde{D}'_i$ and for all $\varphi \in \Phi$

$$\inf_{u \in \mathfrak{M}'} \Big| \frac{1}{q_j} \sum_{i=0}^{q_j - 1} \varphi(A_1 \circ \dots \circ A_j \circ \tau^i_{\alpha_j} x) - \int_M \varphi \, du \Big| < j \frac{\epsilon}{K} \tag{12}$$

where $q_j = m_j Q_j q_{j-1}$ with $m_j \in \mathbb{N}$ arbitrary, $\mathfrak{M}'$ is the simplex generated by $\{\mu_1, \dots \mu_n\}$, and K is the number of disks required to cover N.

(2) for all $x \in \bigcup_{i=1}^{j} D'_i$ the orbit of x under $A_1 \circ \dots \circ A_j \circ \tau^i_{\alpha_j}$ is ϵ-dense.

(3) for all $\varphi \in \Phi$ and all i

$$\Big| \int \varphi \circ A_1 \circ \dots \circ A_j \, d\mu_i - \int \varphi \, d\mu_i \Big| < j \frac{\epsilon}{3K^2} \tag{13}$$

(4) $\mu(A_1 \circ \dots \circ A_1 \tilde{N}_i \Delta \tilde{N}_i) < j \frac{\epsilon}{K}$ for $1 \le i \le d$.

6.2.2. *Induction Step.* We need to construct a diffeomorphism A_{j+1} such that the above properties hold with j replaced by $j+1$. Fix m_j, let $\bar{A} := A_1 \circ \dots A_j$, and let $\tilde{\epsilon} := \min\{\epsilon', \frac{\epsilon}{3K^2}\}$.

Apply Lemma 3 to $\tilde{D}_{j+1}$ with $\alpha = \alpha_j$, $\epsilon = \tilde{\epsilon}$, and $\Phi_{j+1} = \{\varphi \circ \bar{A} : \varphi \in \Phi\}$. For all $x \in \tilde{D}_{j+1}$ and $\varphi \in \Phi$ this gives

$$\inf_{\zeta \in \mathfrak{M}_{j+1}} \Big| \frac{1}{q_{j+1}} \sum_{\Pi^x_{j+1}} \varphi(\bar{A} \circ A_{j+1} \circ \tau^i_{\alpha_{j+1}} x) - \frac{\operatorname{Card}(\Pi^x_{j+1})}{q_{j+1}} \int_{\tilde{D}_{j+1}} \varphi \circ \bar{A} \, d\zeta \Big| < \tilde{\epsilon} < \frac{\epsilon}{3K^2} \tag{14}$$

where $\mathfrak{M}_{j+1}$ is the simplex of measures generated by $\{\zeta^{j+1}_1, \dots, \zeta^{j+1}_n\}$. By Property 5 of Lemma 2

$$\Big| \int \varphi \circ \bar{A} \circ A_{j+1} \, d\mu_i - \int \varphi \circ \bar{A} \, d\mu_i \Big| < \frac{\epsilon}{3K^2}.$$

This together with the induction hypothesis, (13), yields

$$\Big| \int \varphi \circ \bar{A} \circ A_{j+1} \, d\mu_i - \int \varphi \, d\mu_i \Big| < (j+1) \frac{\epsilon}{3K^2} < \frac{\epsilon}{3K} \tag{15}$$

as required. Combining (11), (14), and (15) yields

$$\inf_{u \in \mathfrak{M}'} \Big| \frac{1}{q_{j+1}} \sum_{\Pi^x_{j+1}} \varphi(\bar{A} \circ A_{j+1} \circ \tau^i_{\alpha_{j+1}} x)) - \frac{\operatorname{Card}(\Pi^x_{j+1})}{q_{j+1}} \int_M \varphi \, du \Big| < \frac{\epsilon}{K}. \tag{16}$$

For orbits contained in Σ_{j+1}, which includes the orbit of any $x \in D'_{j+1}$, $\operatorname{Card}(\Pi_{j+1}) = q_{j+1}$ the previous estimate reduces to

$$\inf_{u \in \mathfrak{M}'} \Big| \frac{1}{q_{j+1}} \sum_{i=0}^{q_{j+1}-1} \varphi(\bar{A} \circ A_{j+1} \circ \tau^i_{\alpha_{j+1}} x)) - \int_M \varphi \, du \Big| < (j+1) \frac{\epsilon}{K} \tag{17}$$

which is the next step for (12). It remains to prove (17) for points in $\bigcup_{i=1}^{j} \tilde{D}'_i \backslash \tilde{D}'_{j+1}$. Orbits in $\bigcup_{i=1}^{j} \tilde{D}'_i$ but outside of Σ_{j+1} are unaffected by A_{j+1} and hence (17) follows from (12) since m_j was arbitrary. Consider an orbit in $\bigcup_{i=1}^{j} \tilde{D}'_i$ which intersects but is not contained in Σ_{j+1}. By Property 1 of Lemma 3 it follows that the portion of the orbit outside Σ_{j+1} can be decomposed into N τ_{α_j}-orbits where

$$N := \frac{C}{q_j} \qquad\qquad C := \mathrm{Card}\big((\Pi^x_{j+1})^c\big).$$

Applying (12) to one of the τ_{α_j}-orbits produces

$$\inf_{u\in\mathfrak{M}'}\Big|\frac{1}{q_j}\sum_{i=0}^{q_j}\varphi(\bar{A}\circ A_{j+1}\circ\tau^i_{\alpha_j}x_l)-\int_M\varphi\,du\Big|<j\frac{\epsilon}{K}.$$

Multiplying by q_j and dividing by q_{j+1} yields

$$\inf_{u\in\mathfrak{M}'}\Big|\frac{1}{q_{j+1}}\sum_{i=0}^{q_j}\varphi(\bar{A}\circ A_{j+1}\circ\tau^i_{\alpha_j}x_l)-\frac{C}{q_{j+1}}\int_M\varphi\,d\Big(\frac{q_j}{C}u\Big)\Big|<j\frac{\epsilon}{K}\frac{q_j}{q_{j+1}}.$$

Summing over the N orbits and using the convexity of $\mathfrak{M}'$ gives

$$\inf_{u\in\mathfrak{M}'}\Big|\frac{1}{q_{j+1}}\sum_{(\Sigma^x_{j+1})^c}\varphi(\bar{A}\circ A_{j+1}\circ\tau^i_{\alpha_{j+1}}x)-\frac{C}{q_{j+1}}\int_M\varphi\,du\Big|<j\frac{\epsilon}{K}.$$

Adding (16) to this gives (17) as required.

Since A_{j+1} only affects $\tilde{D}_{j+1}$ Lemma 3 gives

$$\mu(A_{j+1}\tilde{N}_i\Delta\tilde{N}_i)<\lambda(A_{j+1}\tilde{C}_i\Delta\tilde{C}_i)<\frac{\epsilon}{K}.$$

By induction hypothesis 4 and the triangle inequality the result follows.

Lemma 3 proves that any orbit contained in Σ_{j+1} is $\epsilon/2$-dense on D_{j+1}; the orbit of any point in $\tilde{D}'_{j+1}$ meets any $\epsilon/2$-ball centered at a point in $\tilde{D}_{j+1}$. By (10) any $\epsilon/2$-ball in M must intersect $\tilde{D}_{j+1}$. Thus any ϵ-ball in M contains an $\epsilon/2$-ball centered at a point in $\tilde{D}_{j+1}$. By the argument used above any other orbit in $\bigcup_{i=1}^j\tilde{D}'_i$ contains a τ_{α_j}-orbit which is unaffected by A_{j+1} and ϵ-dense by induction hypothesis 2.

This completes the proof of the proposition. □

7. The Proof of the Theorem

The proof follows from induction using Proposition 4.

Lemma 5. *Let $T=h\circ\tau_\alpha\circ h^{-1}$ where h is a smooth measure preserving diffeomorphism of M and $\alpha\in\mathbb{Q}$. Let Φ be a finite set of continuous functions on M. For all $\epsilon>0$ there exists a smooth measure preserving diffeomorphism of M, A, and $\alpha'=p'/q'$ with $(p',q')=1$ such that*

$$T':=h\circ A\circ\tau_{\alpha'}\circ A^{-1}\circ h^{-1}$$

satisfies

(1) $d_{C^\infty}(T,T')<\epsilon$
(2) $d_q(T,T'):=\max_{x\in M}\max_{0\le i\le q}d(T^ix,(T')^ix)<\epsilon$
(3) $\mu(h\circ A\,\tilde{N}_i\Delta h\tilde{N}_i)<\epsilon$
(4) *for all* $\varphi\in\Phi$

$$\min_{\xi\in\Xi}\Big|\frac{1}{q'}\sum_{i=0}^{q'-1}\varphi((T')^ix)-\int\varphi d\xi\Big|<\epsilon$$

where Ξ *is the simplex generated by* $\{\xi_1, \dots, \xi_n\}$ *where* $\xi_i = h^* \mu_i$.

Proof. Apply Proposition 4 to τ_α with the set of functions $\{\varphi \circ h : \varphi \in \Phi\}$ to get A. Define α' by

$$\alpha' := \alpha + \beta \qquad\qquad \beta := \frac{1}{mQq}$$

where$m$ is arbitrary, Q is given by Proposition 4, and q is the denominator of α. Observe

$$\begin{aligned} T' &= h \circ A \circ \tau_{\alpha'} \circ A^{-1} \circ h^{-1} \\ &= h \circ A \circ \tau_\alpha \circ \tau_\beta \circ A^{-1} \circ h^{-1} \\ &= h \circ \tau_\alpha \circ A \circ \tau_\beta \circ A^{-1} \circ h^{-1} \end{aligned}$$

using the fact that A commutes with τ_α. Now as $m \to \infty$ by definition $\beta \to 0$ and hence $A \circ \tau_\beta \circ A^{-1} \to \mathrm{Id}$ smoothly. Therefore we can find m large enough that

$$d_{C^\infty}(T, T') < \epsilon \quad \text{and} \quad d_q(T, T') < \epsilon.$$

Furthermore, directly from Proposition 4

$$\mu(A\tilde{N}_i \Delta \tilde{N}_i) = \mu(h \circ A\tilde{N}_i \Delta h\tilde{N}_i) < \epsilon.$$

Similarly Proposition 4 generates the following inequality

$$\min_{u \in \mathfrak{M}'} \left| \frac{1}{q'} \sum_{i=0}^{q'-1} \varphi(h \circ A \circ \tau_{\alpha'}^i x) - \int \varphi \circ h \, du \right| < \epsilon.$$

Replacing x by $A^{-1} \circ h^{-1} x$ we get the desired property. □

Proof of the Theorem. Let $\{\epsilon_n\}$ be a summable monotone decreasing sequence and $\{\varphi_n\}$ a dense set of continuous functions. Applying the previous lemma inductively we can produce a sequence of numbers $\{\alpha_n\}$ and transformations $\{h_n\}$ such that the sequence of induced transformations $\{T_n\}$ satisfies

(1) $d_{C^\infty}(T_n, T_{n+1}) < \epsilon_{n+1}$
(2) $d_{q_n}(T_n, T_{n+1}) < \epsilon_{n+1}$ where q_n is the period of T_n
(3) $\mu(h_{n+1}\tilde{N}_i \Delta h_n \tilde{N}_i) < \epsilon_{n+1}$
(4) for $\varphi \in \{\varphi_1, \dots, \varphi_n\}$

$$\min_{\xi^n \in \Xi_n} \left| \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}-1} \varphi(T_{n+1}^i x) - \int \varphi \, d\xi^n \right| < \epsilon_{n+1}$$

where Ξ_n is the simplex generated by $\{\xi_1^n, \dots \xi_d^n\}$

The first condition guarantees that the sequence $\{T_n\}$ has a limit T. T is a C^∞ diffeomorphism.

The second condition means that the sequence $\{h_n \tilde{N}_i\}_{n=1}^\infty$ is a Cauchy sequence in the metric on the associated measure algebra. Since this space is complete there is a limit in the measure algebra.

$$\lim_{n\to\infty} h_n \tilde{N}_i = B_i$$

As a consequence we have weak convergence of the measures $\{\xi_i^n\}$,

$$\lim_{n\to\infty} \xi_i^n = \xi_i$$

where

$$\xi_i(A) = \frac{\mu(A \cap B_i)}{\mu(B_i)}.$$

Notice that, as h_n is measure-preserving, $\mu(B_i) = \mu(\tilde{N}_i)$. Notice also that convergence in the measure algebra is not adapted to the original topology. Although the support of ξ_i^n is $h_n\tilde{N}_i$ the limiting measure ξ_i is supported on the whole space. This is another manifestation of the disjunction between the topological setting and the measure-theoretic. Although measure-theoretically B_i is the limit of $h_n\tilde{N}_i$ the first is a dense set while the sets in the sequence are all closed proper subsets. Since the sets $\tilde{N}_i$ and $\tilde{N}_j$ are measurably disjoint so are the limits B_i and B_j.

The measures ξ_i are invariant under the limiting diffeomorphism T. Using the triangle inequality and the invariance of ξ_i^j under T_n

$$\xi_i = \lim_{n\to\infty} \xi_i^n = \lim_{n\to\infty} T_n^*\xi_i^n = T^*\xi_i.$$

It remains to establish that the measures ξ_i exhaust all invariant ergodic probability measures for T. Suppose there exist an invariant measure ζ which is not in the simplex determined by $\{\xi_i\}$. Without loss of generality we may assume that ζ is extremal, that is ζ is an ergodic invariant measure. By the Birkhoff Ergodic Theorem for $\varphi \in L^1(\zeta)$ and for ζ-almost every x

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(T^i x) = \int \varphi\, d\zeta.$$

Since $\{\varphi_i\}$ is dense an approximation argument shows that for any continuous function φ and any x in a set of ζ-full measure

$$\int \varphi\, d\zeta = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(T^i x) = \lim_{n\to\infty} \frac{1}{q_n}\sum_{i=0}^{q_n-1} \varphi(T_n^i x) = \lim_{n\to\infty} \int \varphi\, d\xi^n$$

where ξ^n is in the simplex given by $\{\xi_i^n\}$. Furthermore the sequence of measures $\{\xi^n\}$ depends only on the point x and not on the function φ. Thus

$$\lim_{n\to\infty} \int \varphi\, d\xi^n = \int \varphi\, d\zeta$$

but this means that ζ is in the simplex generated by $\{\xi_i\}$ since this simplex is weakly closed. Contradiction.

This proves that the only possible ergodic measures are $\{\xi_i\}$. Since the sets $\{B_i\}_{i=1}^d$ are measurably disjoint the measures ξ_i are linearly independent. However any non-ergodic measure can be written as a linear combination of ergodic measures so all the ξ_i are ergodic. □

8. Further Results

The method used here to establish the result where d is finite also suffices to produce a diffeomorphism with a countable number of ergodic components. The only differences in the argument occur in the construction of the domains $\{N_i\}$ and in the induction step. We define the sets $\{N_i\}$ given by the Legerdemain Lemma by embedding one disk $d_1 : B(1) \to N$ and defining

$$N_i = \begin{cases} d_1\big(B(1)\big)^c & i = 1 \\ d_1\big(A(\frac{1}{j+1}, \frac{1}{j})\big) & i > 1 \end{cases}$$

where $A(r_1, r_2) = \{x : r_1 \leq \|x\| \leq r_2\}$. On the n-th step of the induction we will take the decomposition of N by $\{N_i'\}_{i=1}^n$ given by

$$N_i' = \begin{cases} N_i & i < n \\ \bigcup_{j=n}^{\infty} N_j & i = n. \end{cases}$$

The permutation α is defined and applied as before but the permutation β is taken to be the identity permutation on parallelepipeds associated to N_n'.

The argument to produce a diffeomorphism which is minimal but which has power of continuum ergodic measures is the simplest of the arguments. In this case we can dispense with the permutation β altogether. Instead of β we apply the identity. The singular measures on the fibers converge to the required ergodic measures.

Though the theorem here produces ergodic measures which are absolutely continuous it is possible to modify the construction to give a mixture of absolutely continuous and singular measures. To do this the diffeomorphisms A_n must be taken non-measure-preserving. Constructing the required A_n for this case can be envisioned as a two step process whereby we first construct a measure-preserving diffeomorphism exactly as was done in this paper and then compose it with a diffeomorphism which takes the cylinder $\tilde{N}_i$ to a smaller cylinder contained within it. This smaller cylinder will serve as the $\tilde{N}_i$ for the next step. This diffeomorphism can be chosen to be the identity on $\tilde{N}_j$ for $1 \leq j \leq d-1$ with $i \neq j$. If the measure of the cylinder $\tilde{N}_i$ becomes 0 in the limit then the limiting measure ξ_i will be singular.

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