LIPSCHITZ $p$-INTEGRAL OPERATORS AND LIPSCHITZ $p$-NUCLEAR OPERATORS

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Abstract. In this note we introduce strongly Lipschitz $p$-integral operators, strongly Lipschitz $p$-nuclear operators and Lipschitz $p$-nuclear operators. It is shown that for a linear operator, the Lipschitz $p$-nuclear norm is the same as its usual $p$-nuclear norm under certain conditions. We also prove that the Lipschitz $2$-dominated operators and the strongly Lipschitz $2$-integral operators are the same with equal norms. Finally, we show that the Lipschitz $p$-integral norm of a Lipschitz map from a finite metric space into a Banach space is the same as its Lipschitz $p$-nuclear norm.

1. Introduction

The linear theory of $p$-summing operators goes back to the work of Grothendieck in [G]. And the most remarkable result can date its beginning in the paper of Pietsch [P1]. A. Pietsch defined the class of $p$-summing operators and proved some good properties that made it appear like an interesting class. Among the main results that appear in [P1], we can find the Domination/Factorization Theorem, ideal, inclusion and composition properties and good relations with other classes of linear operators, such as the nuclear and Hilbert-Schmidt operators.

At the beginning of the 80’s, mainly due to [P2], the idea of generalizing the theory of ideals of linear operators to the multilinear (and polynomial) setting appeared.

Recently Farmer and Johnson [FJ] introduced the notion of Lipschitz $p$-summing operators and the notion of Lipschitz $p$-integral operators and proved a nonlinear version of the Pietsch domination theorem. In [CZh1], the authors proved a nonlinear version of Grothendieck’s theorem for Lipschitz $p$-summing operators. However there is no known nonlinear analogue of $p$-nuclear operators.

In this paper, we introduce natural notions of Lipschitz $p$-nuclear operators and strongly Lipschitz $p$-nuclear operators. We also introduce notions of Lipschitz $p$-dominated operators and strongly Lipschitz $p$-integral operators.

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In Section 2, it is shown that for a bounded linear operator $T$ from a separable Banach space into a dual space, the Lipschitz $p$-nuclear norm of $T$ coincides with its $p$-nuclear norm. So the notion of Lipschitz $p$-nuclear operators is really a generalization of $p$-nuclear operators in this setting. In the same section, we also prove a factorization theorem for strongly Lipschitz $p$-nuclear operators and the result is a nonlinear analogue of the factorization theorem for $p$-nuclear operators.

In Section 3, we give a characterization of Lipschitz $p$-dominated operators in terms of domination and factorization. As a consequence we show that a Lipschitz map is Lipschitz 2-dominated if and only if it is strongly Lipschitz 2-integral. Moreover the Lipschitz 2-dominated norm is the same as the strongly Lipschitz 2-integral norm.

In the last section, we provide some applications to Lipschitz maps from finite metric spaces to Banach spaces. To be more precise we show that a Lipschitz $p$-integral operator $T$ from a finite metric space into a Banach space is automatically Lipschitz $p$-nuclear and the Lipschitz $p$-integral norm equals to the Lipschitz $p$-nuclear norm. We also give some estimates of the norms in terms of the Lipschitz constant of $T$ and the cardinality of the finite metric space. We believe that the natural notions introduced together with the results obtained are pioneering and provide foundational contribution to the theory of nonlinear $p$-integral and nonlinear $p$-nuclear operators.

We assume readers are familiar with linear $p$-summing, $p$-integral and $p$-nuclear operators. Standard notations and related results for linear operators can be found in [DJT],[P] and [T].

2. Nonlinear $p$-nuclear operators

Suppose that $1 \leq p < \infty$ and that $u : X \to Y$ is a linear operator between Banach spaces. We say that $u$ is $p$-summing if there is a constant $c \geq 0$ such that regardless of the natural number $n$ and regardless of the choice of $x_1, \ldots, x_n$ in $X$ we have

$$\left( \sum_{i=1}^{n} \|u(x_i)\|^p \right)^{1/p} \leq c \cdot \sup \left\{ \left( \sum_{i=1}^{n} |x^*(x_i)|^p \right)^{1/p} : x^* \in B_{X^*} \right\},$$

where $B_{X^*}$ is the unit ball of $X^*$. The least constant $c$ for which this inequality always holds is denoted by $\pi_p(u)$. We shall write $\Pi_p(X,Y)$ for the set of all $p$-summing operators from $X$ to $Y$.

We recall that a linear operator $u : X \to Y$ between Banach spaces is $p$-nuclear($1 \leq p \leq \infty$) if there are two bounded linear operators $a : \ell_p \to Y$ and $b : X \to \ell_\infty$ and $\lambda \in \ell_p$ such
that the following diagram commutes:

\[
\begin{array}{c}
X \xrightarrow{u} Y \\
\downarrow b \\
\ell_\infty \xrightarrow{M_\lambda} \ell_p
\end{array}
\]

where \( M_\lambda : \ell_\infty \rightarrow \ell_p \) is the diagonal operator defined as follows: \( M_\lambda((\xi_n)_n) = (\lambda_n \xi_n)_n \), \((\xi_n)_n \in \ell_\infty \). Then \( \|M_\lambda\| = \pi_p(M_\lambda) = t_p(M_\lambda) = \|\lambda\|_p \). The collection of all \( p \)-nuclear operators between \( X \) and \( Y \) is denoted by \( \mathcal{N}_p(X,Y) \). For \( u \in \mathcal{N}_p(X,Y) \), we define the \( p \)-nuclear norm \( \nu_p(u) \) of \( u \) to be the infimum of \( \|a\| \cdot \|M_\lambda\| \cdot \|b\| \), the infimum being taken over all factorizations as above. It is well-known (see, for example, Proposition 8.11 in [T]) that \( u : X \rightarrow Y \) is \( p \)-nuclear (\( 1 \leq p \leq \infty \)) if and only if \( u \) can be written in the form \( u = \sum_j x_j^* \otimes y_j \), where \((x_j^*)_j \in X^* \) and \((y_j)_j \in Y \) satisfy \( N_p((x_j^*)_j,(y_j)_j) < \infty \). Here

\[
\nu_p((x_j^*)_j,(y_j)_j) = \left( \sum_j \|x_j^*\| \right) \left( \sup_j \|y_j\| \right), \quad p = 1
\]

\[
N_p((x_j^*)_j,(y_j)_j) = \left( \sum_j \|x_j^*\|^{\frac{1}{p}} \right) \sup_{y^* \in B_{Y^*}} \left( \sum_j |<y^*, y_j>|^p \right)^{\frac{1}{p}}, \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p^*} = 1
\]

Moreover, \( \nu_p(u) = \inf N_p((x_j^*)_j,(y_j)_j) \), the infimum being taken over all such representations as above.

Let \( X \) be a metric space and \( Y \) be a Banach space. We say that a Lipschitz mapping \( T : X \rightarrow Y \) is called Lipschitz \( p \)-nuclear if there are two Lipschitz mappings \( A : \ell_p \rightarrow Y \) and \( B : X \rightarrow \ell_\infty \) and \( \lambda \in \ell_p \) such that the following diagram commutes:

\[
\begin{array}{c}
X \xrightarrow{T} Y \\
\downarrow B \\
\ell_\infty \xrightarrow{M_\lambda} \ell_p
\end{array}
\]

We set \( \nu_L^p(T) := \inf \text{Lip}(A) \cdot \|M_\lambda\| \cdot \text{Lip}(B) \), the infimum being taken over all factorizations as above. The collection of all Lipschitz \( p \)-nuclear operators from \( X \) to \( Y \) is denoted by \( \mathcal{N}_L^p(X,Y) \).

Next theorem shows that under certain conditions, the Lipschitz \( p \)-nuclear norm of a Lipschitz mapping coincides with its \( p \)-nuclear norm. Differentiation technique is used in the proof. For definitions of Gâteaux differentiability, \( \omega^* \)-Gâteaux differentiability, Gauss null sets and Aronszajn null sets, we refer readers to the book of Benyamini and Lindenstrauss [BL].
Theorem 2.1. Let $T$ be a bounded linear operator from a separable Banach space $X$ into a dual space $Y$. Then $\nu_p^L(T) = \nu_p(T)(1 \leq p < \infty)$.

Proof. We use the method of [FJ] and [JMS]. Consider a typical Lipschitz $p$-nuclear factorization:

$$
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow \quad \quad B & \quad & \quad \uparrow \quad \quad A \\
\ell_\infty & \xrightarrow{M_\lambda} & \ell_p
\end{array}
$$

It follows from Theorem 7.9 ([BL]) that $B : X \to \ell_\infty$ is $\omega^*$-Gâteaux differentiable outside a Gauss null set. Take such $x_0 \in X$. Then

$$
D_{B^*}(x_0)(x) = \omega^* - \lim_{t \to 0} \frac{B(x_0 + tx) - B(x_0)}{t}
$$

exists for all $x \in X$ and is a bounded linear operator in $x$. Since $X$ is separable, it follows from Theorem 6.42 ([BL]) that $M_\lambda B : X \to \ell_p$ is Gâteaux differentiable outside an Aronszajn null set. It is well-known (see, Proposition 6.25, Proposition 6.27 and Theorem 6.32 of [BL]) that Gauss null sets coincide with Aronszajn null sets. Since $M_\lambda : \ell_\infty \to \ell_p$ is weak*-to-weak continuous, we have

$$
M_\lambda D_{B^*}(x_0) = D_{M_\lambda B}(x_0), x_0 \in X \setminus G, G \text{ is a Gauss null set in } X.
$$

By translation, we may assume that $x_0 = 0$ and $B(0) = 0$, that is, $M_\lambda D_{B^*}(0) = D_{M_\lambda B}(0)$.

Next we show that $B$ can be replaced by $D_{B^*}(0)$ by constructing a Lipschitz map $\tilde{B} : \ell_p \to Y$ such that

$$
T = \tilde{B} M_\lambda D_{B^*}(0) \quad \text{and} \quad \text{Lip}(\tilde{B}) \leq \text{Lip}(A).
$$

Indeed, define, for each $n$,

$$
B_n : \ell_p \to Y \quad \text{by} \quad \xi \mapsto nA(\xi/n), \quad \text{for all} \ \xi \in \ell_p.
$$

Then $\text{Lip}(B_n) = \text{Lip}(A)$ and, for all $x \in X$, we have

$$
\|Tx - B_n M_\lambda D_{B^*}(0)(x)\| \leq \text{Lip}(A) \|n M_\lambda B\left(\frac{x}{n}\right)- D_{M_\lambda B}(0)(x)\| \to 0 \quad (n \to \infty).
$$

Since $\{B_n(\xi)\}_{n=1}^\infty$ is bounded in $Y$ and $Y$ is a dual space, the weak* limit of $B_n(\xi)$ through some fixed free ultrafilter $\mathcal{U}$ of natural numbers exists for all $\xi \in \ell_p$. Then, we set

$$
\tilde{B}(\xi) = \omega^* - \lim_{\mathcal{U}} B_n(\xi) \quad \text{for all} \ \xi \in \ell_p.
$$

Thus

$$
T = \tilde{B} M_\lambda D_{B^*}(0) \quad \text{and} \quad \text{Lip}(\tilde{B}) \leq \text{Lip}(A).
$$
Finally, since $\tilde{B}|_{M_\lambda D_B^*(0)(X)}$ is linear and $Y$ is 1-complemented in $Y^\ast\ast$, Theorem 7.2 ([BL]) yields that there is a linear operator $\tilde{A} : \ell_p \to Y$ such that

$$\tilde{A}|_{M_\lambda D_B^*(0)(X)} = \tilde{B}|_{M_\lambda D_B^*(0)(X)} = T \quad \text{and} \quad \|\tilde{A}\| \leq \text{Lip}(\tilde{B}).$$

Therefore,

$$T = \tilde{A}M_\lambda D_B^*(0)(X) \quad \text{and} \quad \nu_p(T) \leq \nu_p^L(T).$$

So,

$$\nu_p(T) = \nu_p^L(T).$$

□

Let $M$ be a metric space. $M^\#$ is the space of all real-valued Lipschitz functions under the (semi)-norm $\text{Lip}(\cdot)$. We follow the usual convention of considering $M$ as a pointed metric space by designating a special point $0 \in M$ and identifying $M^\#$ with the Lipschitz functions on $M$ that are zero at 0. With this convention $(M^\#, \text{Lip}(\cdot))$ is a Banach space and the unit ball $B_M^\#$ of $M^\#$ is a compact Hausdorff space in the topology of pointwise convergence on $M$.

A Lipschitz mapping $T : X \to Y$ from a pointed metric space $X$ to a Banach space $Y$ is called strongly Lipschitz $p$-nuclear if $T$ can be written in the form $T = \sum_j f_j \otimes y_j$, where $(f_j)_j$ in $X^\#$ and $(y_j)_j$ in $Y$ satisfy $N_p^L((f_j)_j, (y_j)_j) < \infty$. Here

$$N_p^L((f_j)_j, (y_j)_j) = \left( \sum_j \text{Lip}(f_j) \right) \left( \sup_j \|y_j\| \right), \quad p = 1$$

$$N_p^L((f_j)_j, (y_j)_j) = \left( \sum_j \text{Lip}(f_j)^p \right)^{1/p} \sup_{y^* \in B_{Y^*}} \left( \sum_j |<y^*, y_j>|^p \right)^{1/p}, \quad 1 < p < \infty$$

$$N_p^L((f_j)_j, (y_j)_j) = \left( \sup_j \text{Lip}(f_j) \right) \sup_{y^* \in B_{Y^*}} \sum_j |<y^*, y_j>|, \lim_{j} \text{Lip}(f_j) = 0, p = \infty.$$

The strongly Lipschitz $p$-nuclear norm is defined by $s\nu_p^L(T) := \inf N_p^L((f_j)_j, (y_j)_j)$, where the infimum being taken over all the strongly Lipschitz $p$-nuclear representations of $T$.

Our next result is a factorization theorem for strongly Lipschitz $p$-nuclear operators.

**Theorem 2.2.** Let $1 \leq p \leq \infty$ and let $X$ be a pointed metric space and $Y$ be a Banach space. A Lipschitz mapping $T : X \to Y$ is strongly Lipschitz $p$-nuclear if and only if $T$ has a factorization $T = AM_\lambda B$ such that the following diagram commutes:

$$\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow B & & \uparrow A \\
\ell_\infty & \xrightarrow{M_\lambda} & \ell_p \quad (c_0, p = \infty)
\end{array}$$
where $B$ is a Lipschitz mapping from $X$ to $\ell_\infty$ with $B(0) = 0$, $M_\lambda \in L(\ell_\infty, \ell_p) (L(\ell_\infty, c_0), p = \infty)$ a diagonal operator and $A \in L(\ell_p, Y) (L(c_0, Y), p = \infty)$. Moreover, $\text{sr}_p^L(T) = \inf \|A\| \cdot \|M_\lambda\| \cdot \text{Lip}(B)$, where the infimum is taken over all the above factorizations.

Proof. Assume that $T$ is strongly Lipschitz $p$-nuclear and let $T = \sum_j f_j \otimes y_j$, with $(f_j)_j$ in $X^*$ and $(y_j)_j$ in $Y$ be a strongly Lipschitz $p$-nuclear representation of $T$. Let

$$B : X \to \ell_\infty, \quad x \mapsto \left( \frac{f_j(x)}{\text{Lip}(f_j)} \right)_j$$

$$M_\lambda : \ell_\infty \to \ell_p(c_0, p = \infty), \quad (t_j)_j \mapsto (\text{Lip}(f_j)t_j)_j, \quad \lambda = (\text{Lip}(f_j))_j$$

$$A : \ell_p(c_0, p = \infty) \to Y, \quad (s_j)_j \mapsto \sum_j s_j y_j.$$

Then $B$ is a Lipschitz mapping from $X$ to $\ell_\infty$ with

$$B(0) = 0, \text{Lip}(B) \leq 1, \quad \text{and} \quad \|M_\lambda\| = \left( \sum_j \text{Lip}(f_j)^p \right)^{\frac{1}{p}} (1 \leq p < \infty).$$

For $p = 1$,

$$\|A((s_j)_j)\| = \sup_{y^* \in B_{Y^*}} | < y^*, \sum_j s_j y_j > | \leq \left( \sum_j |s_j| \right) (\sup_j \|y_j\|).$$

Thus

$$\|A\| \leq \sup_j \|y_j\| \quad \text{for} \quad p = 1.$$

For $1 < p < \infty$,

$$\|A((s_j)_j)\| = \sup_{y^* \in B_{Y^*}} | < y^*, \sum_j s_j y_j > | \leq \sup_{y^* \in B_{Y^*}} \left( \sum_j |s_j|^p \right)^{\frac{1}{p}} \left( \sum_j | < y^*, y_j | |p^* \right)^{\frac{1}{p^*}}.$$

Thus

$$\|A\| \leq \sup_{y^* \in B_{Y^*}} \left( \sum_j | < y^*, y_j | \|p^* \right)^{\frac{1}{p^*}} \quad \text{for} \quad 1 < p < \infty.$$

For $p = \infty$,

$$\|M_\lambda\| = \sup_j \text{Lip}(f_j), \quad \|A\| \leq \sup_{y^* \in B_{Y^*}} \sum_j | < y^*, y_j | |.$$

Thus,

$$T = A M_\lambda B \quad \text{and} \quad \inf \|A\| \|M_\lambda\| \text{Lip}(B) \leq \text{sr}_p^L(T).$$

Conversely, if $M_\lambda = \sum_j \delta_j e_j \otimes e_j$ and $\lambda = (\delta_j)_j \in \ell_p, c_0, p = \infty$, then for $x \in X$,

$$T(x) = A M_\lambda B(x) = \sum_j \delta_j < B(x), e_j > A(e_j).$$

Let $f_j = \delta_j < B(\cdot), e_j >$. Then

$$f_j \in X^* \quad \text{and} \quad T = \sum_j f_j \otimes A(e_j).$$
For $p = 1$, 
\[ \sum_j \text{Lip}(f_j) \leq \text{Lip}(B) \|M\| \quad \text{and} \quad \sup_j \|A(e_j)\| \leq \|A\| \]

For $1 < p < \infty$, 
\[ (\sum_j \text{Lip}(f_j)^p)^{\frac{1}{p}} \leq \text{Lip}(B) \|M\| \]

and 
\[ \sup_{y^* \in B_{Y^*}} (\sum_j |< y^*, A(e_j) >|^p)^{\frac{1}{p}} = \sup_{y^* \in B_{Y^*}} (\sum_j |< A^* y^*, e_j >|^p)^{\frac{1}{p}} = \sup_{y^* \in B_{Y^*}} \|A^* y^*\| = \|A\|. \]

For $p = \infty$, 
\[ \|M\| = \sup_j |\delta_j|, \sup_j \text{Lip}(f_j) \leq \text{Lip}(B) \sup_j |\delta_j| \]

and 
\[ \lim_j \text{Lip}(f_j) = 0. \]

Moreover, 
\[ \sup_{y^* \in B_{Y^*}} \sum_j |< y^*, A(e_j) >| = \sup_{y^* \in B_{Y^*}} \sum_j |< A^* y^*, e_j >| = \sup_{y^* \in B_{Y^*}} \|A^* y^*\| = \|A\|. \]

Thus, for $1 \leq p \leq \infty$, 
\[ N_p^L((f_j), (y_j)) \leq \|A\| \|M\| \text{Lip}(B). \]

This implies that 
\[ s\nu_p^L(T) \leq \inf \|A\| \|M\| \text{Lip}(B). \]

**Remark 2.1.** It follows from Theorem 2.2 that strongly Lipschitz $p$-nuclear operators are Lipschitz $p$-nuclear $(1 \leq p < \infty)$.

Using the result of Theorem 2.2 and a similar argument as in Theorem 2.1, we can show that for a linear operator, the strongly Lipschitz $p$-nuclear norm is the same as its usual $p$-nuclear norm if the domain is separable.

**Theorem 2.3.** Let $T$ be a bounded linear operator from a separable Banach space $X$ into a Banach space $Y$. Then $s\nu_p^L(T) = \nu_p(T) \ (1 \leq p < \infty)$. 
3. NONLINEAR p-INTEGRAL OPERATORS

We begin this section with the definitions of Lipschitz $p$-summing operators and Lipschitz $p$-integral operators [FJ] and some known results. Recall that a Lipschitz map $T$ from a metric space $(X, d_X)$ into a metric space $(Y, d_Y)$ is said to be Lipschitz $p$-summing ($1 \leq p < \infty$) if there is a constant $C > 0$ such that regardless of the natural number $n$ and regardless of the choices of $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ in $X$, we have

$$\left( \sum_{i=1}^n d_Y(Tx_i, Ty_i)^p \right)^{1/p} \leq C \sup_{f \in B_X^*} \left( \sum_{i=1}^n |f(x_i) - f(y_i)|^p \right)^{1/p},$$

the least constant $C$ for which the above inequality holds is denoted by $\pi^L_p(T)$. We use $\pi^L_p(X,Y)$ to denote the set of all Lipschitz $p$-summing mappings from $X$ into $Y$. Farmer and Johnson proved in [FJ] a nonlinear Pietsch domination theorem: a map $T : X \to Y$ between two metric spaces $X$ and $Y$ is Lipschitz $p$-summing if and only if there exist a constant $C > 0$ and a regular Borel probability measure $\mu$ on $B_X^*$ such that $\|Tx -Ty\|_p \leq C \int_{B_X^*} |f(x) - f(y)|^p d\mu(f), \forall x, y \in X$

in this case, $\pi^L_p(T) = \inf C$, the infimum being taken over all possible $C$'s and $\mu$'s.

Let $X$ be a metric space and $Y$ be a pointed metric space. We say that a Lipschitz mapping $T : X \to Y$ is Lipschitz $p$-integral ($1 \leq p \leq \infty$) if there are a probability measure space $(\Omega, \Sigma, \mu)$ and two Lipschitz mappings $A : L_p(\mu) \to (Y^#)^*$ and $B : X \to L_\infty(\mu)$ giving rise to the following commutative diagram:

$$\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow & & \downarrow \text{K}_Y \\
L_\infty(\mu) & \xrightarrow{i_p} & L_p(\mu)
\end{array}$$

where $K_Y : Y \to (Y^#)^*$ is the canonical evaluation map. We define the Lipschitz $p$-integral norm $\iota^L_p(T)$ of $T$ to be the infimum of $\text{Lip}(A) \text{Lip}(B)$, the infimum being taken over all such factorizations. The collection of all Lipschitz $p$-integral operators from $X$ to $Y$ is denoted by $\mathcal{I}^L_p(X,Y)$. When $Y$ is a normed space, we can replace $K_Y$ by the canonical embedding $k_Y$ of $Y$ into $Y^{**}$ in the above definition without changing the Lipschitz $p$-integral norm because $Y^{**}$ is norm-one complemented in $(Y^#)^*$ ([L]). An easy check shows that $\mathcal{I}^L_p(X,Y)$ is a Banach space whenever $X$ and $Y$ are Banach spaces. Since the Banach ideal $[\mathcal{I}_p, \iota_p]$ of $p$-integral operators ($1 \leq p \leq \infty$) is maximal, the Lipschitz $p$-integral norm is the same as the usual $p$-integral norm for a linear operator.
Next we introduce the notions of Lipschitz $p$-dominated operators and strongly Lipschitz $p$-integral operators. A Lipschitz mapping $T : X \to Y$ between Banach spaces $X$ and $Y$ is Lipschitz $p$-dominated $(1 \leq p < \infty)$ if there exist a Banach space $Z$ and a $L \in \Pi_p(X, Z)$ such that
\[
\|Tx - Ty\| \leq \|Lx - Ly\|, \forall x, y \in X.
\]
The collection of all Lipschitz $p$-dominated operators between $X$ and $Y$ is denoted by $\mathcal{D}_p^L(X, Y)$. For $T \in \mathcal{D}_p^L(X, Y)$, we set $d_p^L(T)$ to be the infimum of $\pi_p(L)$, the infimum being taken over all the above $Z$ and $L$. We say that a Lipschitz mapping $T : X \to Y$ between Banach spaces $X$ and $Y$ is strongly Lipschitz $p$-integral $(1 \leq p \leq \infty)$ if there are a probability measure space $(\Omega, \Sigma, \mu)$ and a Lipschitz mapping $A : L_p(\mu) \to Y^{**}$ and a bounded linear operator $B : X \to L_\infty(\mu)$ giving rise to the following commutative diagram:
\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{B} & & \downarrow{\iota_Y} \\
L_\infty(\mu) & \xrightarrow{i_p} & L_p(\mu)
\end{array}
\]
We define the strongly Lipschitz $p$-integral norm $s\iota_p^L(T)$ of $T$ to be the infimum of $\text{Lip}(A) \cdot \|B\|$, the infimum being taken over all such factorizations. In this note, we prove that the Lipschitz 2-dominated operators and the strongly Lipschitz 2-integral operators are the same with equal norms. Moreover, it is proved that the Lipschitz $p$-dominated operators and the strongly Lipschitz $p$-integral operators are also the same with equal norms when the domain space is $C(K)$ ($K$ is a compact Hausdorff space).

We give a characterization of strongly Lipschitz $p$-integral operators whose proof is very similar to the proof in the linear setting (see e.g. page 98 in [DJT]). For convenience of readers, we include the proof here.

Throughout the rest of this section, we assume that $X$ and $Y$ are Banach spaces.

**Theorem 3.1.** Let $1 \leq p \leq \infty$. A Lipschitz mapping $T : X \to Y$ is strongly Lipschitz $p$-integral if and only if every time we take a weak$^*$-compact norming subset $K$ of $B_{X^*}$, there exist a regular Borel probability measure $\mu$ on $K$ and a Lipschitz map $\tilde{T} : L_p(\mu) \to Y^{**}$ such that the following diagram commutes:
\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{\iota_X} & & \downarrow{\tilde{T}} \\
C(K) & \xrightarrow{j_p} & L_p(\mu),
\end{array}
\]
where $j_p, i_X$ are the canonical maps. In this case, $s\iota_p^L(T) = \inf \text{Lip}(\tilde{T})$, where the infimum is taken over all possible $\mu$’s and $\tilde{T}$’s.
Proof. We write $\triangle = \inf \text{Lip}(\tilde{T})$, where the infimum is taking over all $\tilde{T}$ for which the diagram in the statement of the theorem commutes. Fix a weak*-compact norming subset $K$ of $B_{X^*}$. Then there exist a regular Borel probability measure $\mu$ on $K$ and a Lipschitz map $\tilde{T} : L_p(\mu) \to Y^{**}$ such that

$$J_Y T = \tilde{T} j_p i_X : X \xrightarrow{i_X} C(K) \xrightarrow{j_p} L_p(\mu) \xrightarrow{\tilde{T}} Y^{**}.$$ 

Note that

$$j_p = i_p j_\infty : C(K) \xrightarrow{j_\infty} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu).$$

Let $B = j_\infty i_X$ and $A = \tilde{T}$. Then

$$J_Y T = Ai_p B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} Y^{**}.$$ 

This implies that

$$s_{L_p}^T(T) \leq \|B\| \text{Lip}(A) \leq \text{Lip}(\tilde{T})$$

and hence $s_{L_p}^T(T) \leq \triangle$.

Conversely, fix a weak*-compact norming subset $K$ of $B_{X^*}$ and a typical factorization

$$J_Y T = Ai_p B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} Y^{**}.$$ 

Since $L_\infty(\mu)$ is injective, there is a bounded linear operator $\tilde{B} : C(K) \to L_\infty(\mu)$ such that

$$\tilde{B} i_X = B \quad \text{and} \quad \|B\| = \|\tilde{B}\|.$$ 

Applying $i_p \tilde{B} : C(K) \to L_p(\mu)$ to Corollary 2.15 in [DJT], we obtain a regular Borel probability measure $\nu$ on $K$ and a bounded linear operator $C : L_p(\nu) \to L_p(\mu)$ such that

$$i_p \tilde{B} = C j_p : C(K) \xrightarrow{j_p} L_p(\nu) \xrightarrow{C} L_p(\mu) \quad \text{and} \quad \|C\| = \pi_p(i_p \tilde{B}).$$

We set $\tilde{T} = AC$ and then

$$J_Y T = Ai_p B = A(i_p \tilde{B}) i_X = A(C j_p) i_X = \tilde{T} j_p i_X.$$ 

Thus

$$\triangle \leq \text{Lip}(\tilde{T}) \leq \text{Lip}(A) \|C\|$$

$$= \text{Lip}(A) \pi_p(i_p \tilde{B})$$

$$\leq \text{Lip}(A) \|\tilde{B}\|$$

$$= \text{Lip}(A) \|B\|.$$ 

This implies that

$$\triangle \leq s_{L_p}^T(T).$$

$\square$
The following result characterizes Lipschitz \( p \)-dominated operators.

**Theorem 3.2.** For a Lipschitz mapping \( T : X \to Y \) the following are equivalent:

(i) \( T \in \mathcal{D}_p^L(X,Y) \);

(ii) there exists a constant \( C > 0 \) such that for any \( \{x_i\}_{i=1}^n \) and \( \{y_i\}_{i=1}^n \) in \( X \), we have

\[
\left( \sum_{i=1}^n \|Tx_i - Ty_i\|^p \right)^{1/p} \leq C \sup_{x \in B_{X^*}} \left( \sum_{i=1}^n |\langle x^*, x_i - y_i \rangle|^p \right)^{1/p};
\]

(iii) there exist a constant \( C > 0 \) and a probability measure \( \mu \) on \( B_{X^*} \) such that for all \( x, y \in X \), we have \( \|Tx - Ty\| \leq C(\int_{B_{X^*}} |\langle x^*, x - y \rangle|^p d\mu(x^*))^{1/p} \). In that case \( d^L_p(T) = \inf \{ C : C \text{ as in (ii)} \} = \inf \{ C : C, \mu \text{ as in (iii)} \} \).

**Proof.** (i) \( \Rightarrow \) (ii). Choose any \( L \in \pi_p(X,Z) \) such that

\[
\|Tx - Ty\| \leq \|Lx - Ly\|, \forall \ x, y \in X.
\]

Then, for any finite sequence \( \{x_i\}_{i=1}^n \) and \( \{y_i\}_{i=1}^n \) in \( X \), we have

\[
\left( \sum_{i=1}^n \|Tx_i - Ty_i\|^p \right)^{1/p} \leq \left( \sum_{i=1}^n \|Lx_i - Ly_i\|^p \right)^{1/p}
\]

\[
\leq \pi_p(L) \sup_{x \in B_{X^*}} \left( \sum_{i=1}^n |\langle x^*, x_i - y_i \rangle|^p \right)^{1/p}.
\]

Thus,

\[
\inf \{ C : C \text{ as in (ii)} \} \leq \pi_p(L).
\]

That is,

\[
\inf \{ C : C \text{ as in (ii)} \} \leq d^L_p(T).
\]

(ii) \( \Rightarrow \) (iii). Fix \( x, y \in X \). Define \( g_{\{x,y\}} : B_{X^*} \to \mathbb{R} \) by \( x^* \mapsto \|Tx - Ty\|^p - C^p|\langle x^*, x - y \rangle|^p \).

Let \( Q \) be the positive convex hull of \( \{g_{\{x,y\}} : x, y \in X\} \), that is,

\[
Q = \{ \sum_i \lambda_ig_{\{x_i,y_i\}} : \sum_i \lambda_i = 1, \lambda_i > 0 \}
\]

and let \( P = \{ F \in C(B_{X^*}, \mathbb{R}) : F(x^*) > 0, \forall x^* \in B_{X^*} \} \).

Then it follows from (ii) and the approximation that \( P \cap Q = \emptyset \). It follows from the separation theorem and the Riesz representation theorem that there is a finite signed Borel regular measure \( \mu \) on \( B_{X^*} \) and a real number \( c \) so that

\[
\int_{B_{X^*}} g d\mu \leq c < \int_{B_{X^*}} F d\mu, \forall \ g \in Q, \forall \ F \in P.
\]
Since $0 \in Q$ and all positive constants are in $P$, we see that $c = 0$ and $\mu$ is a positive measure, which we may assume by rescaling is a probability measure. In particular, for $g_{(x,y)}$, we have

$$\|Tx - Ty\|^p - C^p \int_{B_{X^*}} |\langle x^*, x - y \rangle|^p \, d\mu \leq 0.$$  

Moreover,

$$\inf\{C : C, \mu \text{ as in (iii)}\} \leq \inf\{C : C \text{ as in (ii)}\}.$$  

(iii) $\Rightarrow$ (i). Define

$$L : X \to L_p(B_{X^*}, \mu) \text{ by } x \mapsto (Lx)(x^*) = \langle x^*, x \rangle, \forall x^* \in B_{X^*}.$$  

Then

$$\|Tx - Ty\| \leq C \cdot \|Lx - Ly\| \quad \forall x, y \in X$$

and

$$\pi_p(L) \leq 1.$$  

Thus,

$$d^L_p(T) \leq \pi_p(C \cdot L) \leq C.$$  

This implies

$$d^L_p(T) \leq \inf\{C : C, \mu \text{ as } \text{in (iii)}\}.$$  

Remark 3.1. In the above theorem, $B_{X^*}$ can be replaced by any weak*-compact norming subset of $B_{X^*}$. Theorem 3.2 clearly implies that a Lipschitz $p$-dominated operator is Lipschitz $p$-summing. In the nonlinear Pietsch domination theorem by Farmer and Johnson [FJ], $B_{X^#}$ is used for supremum and integral. However the geometric structure of $X^#$ is poorly understood and in most cases, it is very hard to handle. So the importance of the above theorem is that in (ii) and (iii), we use $B_{X^*}$ for the supremum and integral instead of $B_{X^#}$.

Theorem 3.3. Let $T : X \to Y$ be a Lipschitz mapping and let $K$ be a weak* compact norming subset of $B_{X^*}$. The following are equivalent:

(i) $T \in D^L_p(X,Y)$;

(ii) there exist a regular Borel probability measure $\mu$ on $K$, a closed subspace $X_p$ of $L_p(\mu)$ and a Lipschitz map $\tilde{T} : X_p \to Y$ such that (a) $j_p i_X(X) \subseteq X_p$; (b) $\tilde{T} j_p i_X = T$. In
other words, the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow i_X & & \uparrow \tilde{T} \\
\cap_i X & \xrightarrow{j_{p|X}} & X_p \\
\cap & & \\
C(K) & \xrightarrow{j_p} & L_p(\mu);
\end{array}
\]

(iii) there exist a regular Borel probability measure \( \mu \) on \( K \) and a Lipschitz map \( \tilde{T} : L_p(\mu) \to \ell_\infty(B_Y^*) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow i_X & & \uparrow \tilde{T} \\
C(K) & \xrightarrow{j_p} & L_p(\mu);
\end{array}
\]

(iv) there exist a probability space \((\Omega, \Sigma, \mu)\) and a Lipschitz mapping \( \tilde{u} : L_p(\mu) \to \ell_\infty(B_Y^*) \) and a linear mapping \( v : X \to L_\infty(\mu) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow v & & \uparrow \tilde{u} \\
L_\infty(\mu) & \xrightarrow{i_p} & L_p(\mu);
\end{array}
\]

In addition, we may choose \( \mu \) and \( \tilde{T} \) in (ii) or \( \mu \) and \( \tilde{T} \) in (iii) so that \( \text{Lip}(\tilde{T}) = \text{Lip}(\hat{T}) = d_p^L(T) \); in (iv) we may arrange that \( \|v\| = 1 \) and \( \text{Lip}(\tilde{u}) = d_p^L(T) \).

**Proof.** \((i) \Rightarrow (ii)\): It follows from Theorem 3.2 that there is a regular Borel probability measure \( \mu \) on \( K \) such that for all \( x, y \in X \), we have

\[
\|Tx - Ty\| \leq d_p^L(T) \left( \int_{B_{X^*}} |\langle x^*, x - y \rangle|^p \, d\mu(x^*) \right)^{1/p}.
\]

Define a Lipschitz mapping \( u : j_p i_X(X) \to Y \) by \( u(j_p i_X(x)) = T(x), x \in X \). Then

\[
\text{Lip}(u) \leq d_p^L(T).
\]

We let \( X_p \) be the norm closure of \( j_p i_X(X) \) in \( L_p(\mu) \) and \( \hat{T} \) be the extension of \( u \) to \( X_p \). Then

\[
\text{Lip}(\tilde{T}) \leq d_p^L(T) \quad \text{and} \quad \hat{T} j_p X i_X = T.
\]

For the converse,

\[
d_p^L(T) = d_p^L(\tilde{T} j_p X i_X) \leq \text{Lip}(\tilde{T}).
\]
Thus

\[ d_p^L(T) = \operatorname{Lip}(\hat{T}). \]

(ii) \(\Rightarrow\) (iii): Let \( \mu, X_p \) and \( \hat{T} \) be as in (ii). Since \( l_\infty(BY^\ast) \) is an absolute 1-Lipschitz retract, there is a Lipschitz extension \( \hat{T} \) of \( i_Y\hat{T} \) to \( L_p(\mu) \) with \( \operatorname{Lip}(\hat{T}) = \operatorname{Lip}(i_Y\hat{T}) \). Then \( \hat{T} \) is the required Lipschitz mapping.

(iii) \(\Rightarrow\) (iv): Let \( \mu \) and \( \bar{T} \) be as in (iii). Note that

\[ j_p = i_p j_\infty : C(K) \xrightarrow{j_\infty} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu). \]

Let \( u = j_\infty i_X \). Then

\[ i_Y T = \bar{T} i_p u. \]

Let \( v = \frac{u}{\|u\|} \) and \( \bar{u} = \|u\|\bar{T} \). This implies that

\[ i_Y T = \bar{u} i_p v. \]

Moreover,

\[ \operatorname{Lip}(\bar{u}) \leq \operatorname{Lip}(\bar{T}) = d_p^L(T) \]

and

\[ d_p^L(T) = d_p^L(i_Y T) = d_p^L(\bar{u} i_p v) \leq \operatorname{Lip}(\bar{u}) \pi_p(i_p v) \leq \operatorname{Lip}(\bar{u}). \]

Thus

\[ \|v\| = 1 \text{ and } \operatorname{Lip}(\bar{u}) = d_p^L(T). \]

(iv) \(\Rightarrow\) (i): It follows from Theorem 3.2 and the fact that \( i_p v \) is \( p \)-summing and \( i_Y \) is an isometric embedding.

\[ \square \]

Theorem 3.3 yields the following corollaries immediately.

**Corollary 3.4.** Let \( K \) be a compact Hausdorff space. A Lipschitz mapping \( T : C(K) \to Y \) is Lipschitz \( p \)-dominated if and only if there exist a regular Borel probability measure \( \mu \) on \( K \) and a Lipschitz mapping \( \hat{T} : L_p(\mu) \to Y \) such that \( \hat{T} j_p = T \). Moreover, we may arrange that \( \operatorname{Lip}(\hat{T}) = d_p^L(T) \).
**Corollary 3.5.** A Lipschitz mapping $T : X \to Y$ is Lipschitz 2-dominated if and only if there exist a regular probability measure $\mu$ on $B_{X^*}$ and a Lipschitz mapping $\tilde{T} : \ell_2(\mu) \to Y$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{i_X} & & \downarrow{\tilde{T}} \\
C(B_{X^*}) & \xrightarrow{j_2} & \ell_2(\mu).
\end{array}
$$

Moreover, we may arrange that $\text{Lip}(\tilde{T}) = d_{\ell_2}(T)$.

**Theorem 3.6.** If $T : X \to Y$ is strongly Lipschitz $p$-integral, then it is Lipschitz $p$-dominated with $d_{\ell_p}(T) \leq \text{st}_{\ell_p}(T)$.

**Proof.** Given a typical strongly Lipschitz $p$-integral factorization:

$$
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{B} & & \downarrow{J_Y} \\
L_\infty(\mu) & \xrightarrow{i_p} & \ell_p(\mu).
\end{array}
$$

Since $i_p$ is $p$-summing and $B$ is linear, it follows from the ideal property of Lipschitz $p$-dominated operators that $J_Y T$ is Lipschitz $p$-dominated. Hence $T$ is also Lipschitz $p$-dominated and

$$
d_{\ell_p}(T) = d_{\ell_p}(J_Y T) = d_{\ell_p}(A i_p B) \leq \text{Lip}(A) \|B\|.
$$

This implies

$$
d_{\ell_p}(T) \leq \text{st}_{\ell_p}(T).
$$

\qed

Combining the previous results, we have the following two immediate corollaries.

**Corollary 3.7.** The Lipschitz 2-dominated and strongly Lipschitz 2-integral operators are the same with equality of norms.

**Corollary 3.8.** Let $1 \leq p < \infty$. If $K$ is a compact Hausdorff space, then every Lipschitz $p$-dominated operator $T : C(K) \to Y$ is strongly Lipschitz $p$-integral with equality of norms.
4. Applications to Lipschitz mappings from a finite metric space into a Banach space

This section deals with the relationship between Lipschitz $p$-integral and Lipschitz $p$-nuclear operators from a finite metric space into a Banach space.

**Theorem 4.1.** Let $X$ be a finite metric space and $Y$ be a Banach space. Then, for any $1 \leq p < \infty$ and any mapping $T : X \to Y$, we have

(a) $\nu^L_p(T) = i^L_p(T)$;
(b) $\nu^L_p(T) = i^L_p(T) \leq C \cdot (\log |X|)^2 \cdot \text{Lip}(T)$, where $C$ is an absolute constant.

**Proof.** (a) Consider a typical $p$-integral factorization

$$J_Y T : X \overset{B}{\to} L_{\infty}(\mu) \overset{i_p}{\to} L_p(\mu) \overset{A}{\to} Y^{**}.$$ Fix $\varepsilon > 0$. Then, there exists a finite dimensional subspace $E$ of $L_{\infty}(\mu)$, of dimension $N$ say, together with an isomorphism $\nu : E \to \ell^N_\infty$ such that

$$\|\nu\| \|\nu^{-1}\| \leq 1 + \varepsilon \quad \text{and} \quad B(X) \subseteq E.$$ Moreover, $i_p(E)$ is contained in a $(1 + \varepsilon)$-complemented subspace $F$ of $L_p(\mu)$. Let $J : F \to L_p(\mu)$ be the corresponding canonical embedding and $P : L_p(\mu) \to F$ the corresponding projection with $\|P\| \leq 1 + \varepsilon$. The principle of local reflexivity guarantees the existence of a linear operator $W : \text{span}\{A_i B(X)\} \to Y$ such that

$$\|W\| \leq 1 + \varepsilon \quad \text{and} \quad W|_{\text{span}\{A_i B(X)\} \cap J_Y(Y)} = \text{id}_{J_Y(Y)}.$$ Thus,

$$J_Y T = WJ_Y T.$$ Consider

$$\tilde{u} = Pi_p|_E \nu^{-1} : \ell^N_\infty \to F.$$ Then $\tilde{u}$ is $p$-summing and hence $p$-integral and

$$\pi_p(\tilde{u}) = i_p(\tilde{u}) = \nu_p(\tilde{u}) \leq \|P\| i_p(\nu)|\nu^{-1}\| \leq (1 + \varepsilon)\|\nu^{-1}\|.$$ Note that

$$J_Y T = WJ_Y T = WA_i B = WA J \tilde{u} \nu B.$$ Then
\[
\nu^L_p(T) = \nu^L_p(J_Y T) \\
= \nu^L_p(WAJ\tilde{u}B) \\
\leq \text{Lip}(WAJ)\nu_p(\tilde{u}) \text{Lip}(\nu B) \\
\leq (1 + \varepsilon)^3 \text{Lip}(A) \text{Lip}(B).
\]

Thus
\[
\nu^L_p(T) = \iota^L_p(T).
\]

(b) It follows from Farmer-Johnson Factorization Theorem ([FJ]) that

\[
\begin{align*}
 X & \xrightarrow{T} Y \\
 i_X \downarrow & \uparrow \tilde{u} \\
 i_X(X) & \xrightarrow{j_p|_{i_X(X)}} X_p \\
 \cap & \cap \\
 C(K) & \xrightarrow{j_p} L_p(\mu),
\end{align*}
\]

where \( T = \tilde{u}j_p i_X \) and \( \pi^L_p(T) = \text{Lip}(\tilde{u}) \). It follows from Theorem 1 of [JLS] that there exists \( \hat{u} : L_p(\mu) \rightarrow Y \) such that
\[
\hat{u}|_{X_p} = \tilde{u} \quad \text{and} \quad \text{Lip}(\hat{u}) \leq (C_1 \cdot \log |X|) \text{Lip}(\tilde{u}),
\]

where \( C_1 \) is an absolute constant. Note that
\[
\tilde{u}|_{X_p} = \hat{u} \quad \text{and} \quad \text{Lip}(\hat{u}) \leq (C_1 \cdot \log |X|) \text{Lip}(\tilde{u}),
\]

Let \( b = j_{\infty}i_X \). Then
\[
T = \tilde{u}j_p i_X = \hat{u}i_p b.
\]
Thus
\[
\nu^L_p(T) = \iota^L_p(T) \\
\leq \text{Lip}(b) \cdot \text{Lip}(\hat{u}) \\
\leq (C_1 \cdot \log |X|) \pi^L_p(T).
\]

Finally, as was mentioned in [FJ], J.Bourgain([B]) really proved that
\[
\pi_p^L(id_X) \leq \pi^L_p(id_X) \leq (C_2 \cdot \log |X|),
\]

where \( C_2 \) is an absolute constant. It follows from the ideal property of Lipschitz \( p \)-summing operators that
\[
\pi_p^L(T) = \pi_p^L(T \cdot id_X) \leq (C_2 \cdot \log |X|) \text{Lip}(T).
\]
This implies that
\[ \nu_p^L(T) = \iota_p^L(T) \leq C_1 \cdot C_2 \cdot (\log |X|)^2 \text{Lip}(T). \]

The following result gives a better estimate on the constant of Theorem 4.1 when the range space is a 2-uniformly convex Banach space and the cardinality of \( X \) is relatively small compared to \( p \).

**Theorem 4.2.** Let \( X \) be a finite metric space and \( Y \) be a 2-uniformly convex Banach space. Then, for any \( 2 \leq p < \infty \) and for any mapping \( T : X \to Y \), we have

\[ \nu_p^L(T) \leq 24K_2(Y) \cdot \sqrt{p-1} \cdot \text{Lip}(T)(|X|)^{1/p} \]

where \( K_2(Y) \) is the 2-uniform convexity constant of \( Y \).

**Proof.** Suppose that \( X = \{x_1, \ldots, x_n\} \). Consider the Fréchet map

\[ \mathcal{F} : X \to \ell^n_\infty(n = |X|) \text{ given by } \mathcal{F}(x) = (d(x, x_i))_{i=1}^n. \]

Then \( \mathcal{F} \) is an isometric embedding. Define \( g : \text{Id}_{\ell_\infty^p} \mathcal{F}(X) \to Y \) by \( \text{Id}_{\ell_\infty^p} \mathcal{F}(x) \mapsto T(x) \),

where \( \text{Id}_{\ell_\infty^p} : \ell^n_\infty \to \ell^n_\infty \) and \( \text{Id}_{\ell_p^\infty} : \ell^n_p \to \ell^n_\infty \) are the identity mappings. Note that \( \text{Id}_{\ell_\infty^p} \) is \( p \)-nuclear with \( \nu_p(\text{Id}_{\ell_\infty^p}) = \iota_p(\text{Id}_{\ell_\infty^p}) = \pi_p(\text{Id}_{\ell_\infty^p}) = n^{1/p} \). It follows from Theorem 1.1 ([MN]) that there exists \( \tilde{g} : \ell^n_p \to Y \) such that

\[ \text{Lip}(\tilde{g}) \leq 24K_2(Y) \sqrt{p-1} \text{Lip}(g) \]

\[ \leq 24K_2(Y) \sqrt{p-1} \cdot \text{Lip}(T), \]

and

\[ \tilde{g} \big|_{\text{Id}_{\ell_\infty^p} \mathcal{F}(X)} = g. \]

Then

\[ T = \tilde{g} \circ \text{Id}_{\ell_\infty^p} \circ \mathcal{F}, \]

and

\[ \nu_p^L(T) \leq \text{Lip}(\tilde{g}) \nu_p(\text{Id}_{\ell_\infty^p}) \]

\[ \leq 24K_2(Y) \cdot \sqrt{p-1} \cdot \text{Lip}(T) \cdot n^{1/p}. \]

\[ \square \]
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