GEOMETRIC PROPERTIES ON NON-COMPLETE SPACES

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This paper is dedicated to the beloved memory of Prof. Antonio Aizpuru

Abstract. The purpose of this paper is to study certain geometrical properties for non-complete normed spaces. We show the existence of a non-rotund Banach space with a rotund dense maximal subspace. As a consequence, we prove that every separable Banach space can be renormed to be non-rotund and to contain a dense maximal rotund subspace. We then construct a non-smooth Banach space with a dense maximal smooth subspace. We also study the Krein-Milman Property on non-complete normed spaces and provide a sufficient condition for an infinite dimensional Banach space to have an infinite dimensional, separable quotient.

1. Introduction

Various geometric properties of normed linear spaces have been studied for many years. An important question is to find out which geometric properties of non-complete normed spaces extend to their completion. It is well known [7] that a Banach space is reflexive if and only if every bounded linear functional on $X$ is norm attaining. In the 1960’s, Frank Deutsch asked whether every normed linear space is complete if every bounded linear functional on it is norm attaining. If this was true, then the space is automatically reflexive. However James [7] gave a negative answer by constructing the following example. This is probably the first result on the weakness of geometrical properties under the lack of completeness.

Theorem 1.1 (James, 1971). There exists a non-complete normed space on which every functional is norm-attaining.

In 1976 Blatter toughened the condition that “all functionals are norm-attaining” accomplishing something positive (see [1]).

Theorem 1.2 (Blatter, 1976). Let $X$ be a normed space. If every closed, convex subset of $X$ has a minimum-norm element, then $X$ is complete (and thus reflexive).

Some of the geometric properties pass to dense subspaces. Two of the most important such properties are (uniform) rotundity and (uniform) smoothness. The modulus of rotundity of a normed linear space $X$ is the function $\delta : [0, 2] \to [0, 1]$ defined by $\delta(\epsilon) = \inf \{1 - \frac{\|x+y\|}{2} : x, y \in B_X, \|x - y\| \geq \epsilon\}$, where $B_X$ is the closed unit ball of $X$. If $\delta(\epsilon) > 0$ for all $0 < \epsilon \leq 2$, then $X$ is said to be uniformly rotund. By approximation, it is easy to see from the definition that if a
dense subspace of a Banach space is uniformly rotund (uniformly smooth), then the Banach space itself is also uniformly rotund (uniformly smooth). So uniform convexity (uniform smoothness) extends from non-complete normed spaces to their completion. Considering the above discussion, a natural question we have here is that whether rotundity and smoothness extend from non-complete normed spaces to their completion. It is quite surprising that the answer turns out to be negative.

In Section 2, we construct a non-complete normed space $X$ so that $X$ is rotund but the completion $\overline{X}$ of $X$ is non-rotund. Moreover, $X$ as a dense subspace of $\overline{X}$ is also maximal. As a consequence, we have the following theorem.

**Theorem A.** Let $X$ be an infinite dimensional Banach space that admits a rotund equivalent norm. Then, $X$ admits an equivalent renorming such that $X$ is non-rotund but has a rotund, dense, and maximal subspace.

A subspace of a Banach space is said to be maximal if it is of codimension one. Since every separable Banach space can be renormed to be rotund by Clarkson [3], we get the following corollary immediately.

**Corollary A.** Every infinite dimensional separable Banach space can be renormed to be non-rotund and to contain a rotund, dense, and maximal subspace.

Locally uniformly rotund spaces can be viewed as objects better than rotund spaces but worse than uniformly rotund spaces (the definition is contained in Section 2). Given Theorem A and the discussion above, a natural question is whether there exists a non-rotund Banach space with a locally uniformly rotund dense subspace. Section 3 is devoted to the study of this question. Although we believe that the answer is positive, we only get the following partial result.

**Theorem B.** If there exists a three dimensional normed space so that the unit sphere is symmetric with respect to the $z$-coordinate and contains exactly two symmetric maximal segments, then every infinite dimensional Banach space that admits a locally uniformly rotund norm can be renormed to be non-rotund and to contain a locally uniformly rotund dense maximal subspace.

We also give a closed surface in $\mathbb{R}^3$ which is a possible candidate to satisfy the hypothesis of Theorem B.

In Section 4, we construct a non-complete normed space $Y$ so that $Y$ is Fréchet-smooth but the completion is not Gateaux-smooth. To be more precise, we have

**Theorem C.** Let $Y$ be an infinite dimensional Banach space that admits a Fréchet-smooth equivalent norm. Then $Y$ admits an equivalent renorming such that $Y$ is non-Gateaux smooth but has a Fréchet-smooth, dense, and maximal subspace.

The study of the conditions for the equivalence of RNP and Krein-Milman property (KMP) is an important and interesting subject. There are many positive results for certain classes of Banach spaces. However, it is shown in [10] that RNP and KMP are incomparable properties in non-complete spaces. So an open problem is whether the KMP for a normed space implies the completeness of the normed
space. In Section 5, we show that the KMP can substitute completeness in certain settings. Using Baire Category theorem, it is easy to see that every infinite dimensional Banach space does not admit a countable Hamel basis. By a simple construction, we prove

**Theorem D.** Every infinite dimensional normed space with the KMP does not have a countable Hamel basis.

The failing of KMP in non-complete spaces exists almost everywhere in virtue of the following theorem.

**Theorem E.** Every infinite dimensional separable Banach space has a dense subspace lacking the KMP.

In Section 6, we revisit the characterization of the separable quotient problem by Rosenthal [13, 14]. By introducing the so called cs-complemented subspaces of a Banach space, we release a little bit of the conditions given by Rosenthal.

2. A non-rotund Banach space with a dense and maximal rotund subspace

**Definition 2.1.** Let $X$ be a normed space. Let $x \in S_X$. Then:

1. The point $x$ is said to be a rotund point of $B_X$ if $x$ is not contained in any non-trivial segment of $S_X$, that is, if $y \in S_X$ and $\|\frac{x+y}{2}\| = 1$, then $x = y$. The set of rotund points of $B_X$ is usually denoted as rot $(B_X)$.

2. The point $x$ is said to be a locally uniformly rotund point of $B_X$ if every sequence $(y_n)_{n \in \mathbb{N}} \subset B_X$ such that $(\|\frac{x+y_n}{2}\|)_{n \in \mathbb{N}}$ converges to 1 is convergent to $x$. The set of locally uniformly rotund points of $B_X$ is usually denoted as rot$_u (B_X)$.

As the reader may consult in the previous three references, a normed space is rotund or locally uniformly rotund if every element in its unit sphere is a rotund point or a locally uniformly rotund point. Let $X$ be a normed space. We use $S_X$ to denote its unit sphere and $B_X$ to denote its unit ball.

**Lemma 2.2.** Let $X$ be an infinite dimensional Banach space enjoying the following property: there exist $x \neq y \in S_X$ such that $[x, y]$ and $[-x, -y]$ are the only maximal segments in $S_X$. Then, $X$ is non-rotund but has a rotund dense and maximal subspace.

**Proof.** Let $f : X \to \mathbb{R}$ be a non-continuous linear functional such that $f(x) = 0$ and $f(y) = 1$. We know that ker $(f)$ is a dense and maximal subspace of $X$ and $([x, y] \cup (-x, -y)) \cap$ ker $(f) = \emptyset$. By hypothesis, rot $(B_X) = S_X \setminus ([x, y] \cup [-x, -y])$. We will show that ker $(f)$ is rotund. Observe that

$$S_{\text{ker}(f)} \setminus \{x, -x\} \subseteq S_X \setminus ([x, y] \cup [-x, -y]) \cap \text{ker}(f) = \text{rot} (B_X) \cap \text{ker}(f) \subseteq \text{rot} (B_{\text{ker}(f)}) .$$

It remains to show that $x, -x \in \text{rot} (B_{\text{ker}(f)})$. Assume not. Then, there exists $z \in S_{\text{ker}(f)} \setminus \{x, -x\}$ such that $[z, x] \subset S_{\text{ker}(f)} \subset S_X$ and $[-z, -x] \subset S_{\text{ker}(f)} \subset S_X$. By hypothesis, $z \in [y, x]$ and $-z \in [-y, -x]$, which is a contradiction. □
Remark 2.3. In the proof of the previous lemma it would have been much easier to take \( f : X \to \mathbb{R} \) a non-continuous linear functional such that \( f(x) = f(y) = 1 \). However, we wanted to show that the dense maximal rotund subspace can be chosen to intersect a segment of the superspace. We will show in the next section that this situation is not possible with dense locally uniformly rotund subspaces.

In the next section we will discuss about the existence of infinite dimensional Banach spaces exists enjoying the hypothesis of Lemma 2.2.

Lemma 2.4. There exists a 2-dimensional Banach space \( Y \) with the following property: there exist \( x \neq y \in S_Y \) such that \([x,y]\) and \([-x,-y]\) are the only maximal segments in \( S_X \).

Proof. The union of the two circles
\[
\left\{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 \leq 1 \right\}
\]
and
\[
\left\{(x,y) \in \mathbb{R}^2 : (x+1)^2 + y^2 \leq 1 \right\}
\]
and the unit square
\[
\left\{(x,y) \in \mathbb{R}^2 : |x|,|y| \leq 1 \right\}
\]
determines the unit ball of a 2-dimensional Banach space whose only two maximal segments contained in the unit sphere are \([-1,1),(1,1)\] and \([1,-1),(-1,-1)\]. This norm can be expressed as follows:
\[
\|\langle x, y \rangle\| := \begin{cases} 
\frac{x^2+y^2}{2|x|} & \text{if } |y| < |x| \\
\frac{1}{|y|} & \text{if } |x| \leq |y|
\end{cases}
\]
where \( x, y \in \mathbb{R} \). □

A natural approach is to extend the previous construction of the norm in 2-dimensional spaces to infinite dimensional spaces. The way of extension is expressed in the following remark.

Remark 2.5. If \( \|\cdot\| \) is a norm on \( \mathbb{R}^n \) and \( X_1, \ldots, X_n \) are Banach spaces, then we define the following norm on \( X_1 \times \cdots \times X_n \):
\[
\|(x_1, \ldots, x_n)\| := \|(\|x_1\|, \ldots, \|x_n\|)\|
\]
for all \( (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n \).

Unfortunately, we will prove that this way does not serve our purposes.

Theorem 2.6. Let \( H \) be an infinite dimensional Banach space. Let \( X \) be a closed maximal subspace of \( H \) and let \( y \in S_H \) such that \( X \) is a complement of \( \mathbb{R}y \) in \( H \). Assume that the norm of \( H \) is given by
\[
\|x + \lambda y\| := \begin{cases} 
\frac{\|x\|^2 + \lambda^2}{2\|x\|} & \text{if } |\lambda| < \|x\| \\
\frac{1}{|\lambda|} & \text{if } \|x\| \leq |\lambda|
\end{cases}
\]
where \( x \in X \) and \( \lambda \in \mathbb{R} \). Then, \( H \) does not have dense rotund subspaces.

Proof. Observe that the unit ball of \( H \) has a face with non-empty interior relative to the unit sphere \( \{B_H \times \{y\}\} \), which does not allow any rotund dense subspace. □

Not even switching coordinates we can accomplish what we want.
Theorem 2.7. Let $H$ be an infinite dimensional Hilbert space. Let $x \in S_H$ and $Y$ the orthogonal complement of $\mathbb{R}x$. Consider the equivalent norm on $H$ given by

$$
\|\lambda x + y\| := \begin{cases} 
\frac{\lambda^2 + \|y\|^2}{2|\lambda|} & \text{if } \|y\| < |\lambda| \\
\frac{\|y\|^2}{2|\lambda|} & \text{if } |\lambda| \leq \|y\|
\end{cases}
$$

where $\lambda \in \mathbb{R}$ and $y \in Y$. Let $\lambda, \gamma \in \mathbb{R}$ and $y, z \in Y$ such that

$$
\|\lambda x + y\| = \|\gamma x + z\| = \left\| \frac{\lambda + \gamma}{2} x + \frac{y + z}{2} \right\| = 1.
$$

Then we have two possibilities:

1. $\lambda, \gamma \in [0, 1]$ and $y = z \in S_Y$.
2. $|\lambda| \leq \|y\| = 1$, $\gamma^2 + \|z\|^2 = 2|\gamma|$, and

$$
2\langle y, z \rangle = \begin{cases} 
-\lambda^2 + (4 - 2\gamma)\lambda + (2\gamma - 1) & \text{if } \gamma > 0 \\
-\lambda^2 - (4 + 2\gamma)\lambda - (2\gamma + 1) & \text{if } \gamma < 0.
\end{cases}
$$

As a consequence, $H$ does not admit rotund dense and maximal subspaces under this equivalent norm.

Proof. First off, note that if $\lambda \in \mathbb{R}$ and $y \in Y$ are such that $\|\lambda x + y\| = 1$, then either $|\lambda| \leq \|y\| = 1$ or $\lambda^2 + \|y\|^2 = 2|\lambda|$ and thus $2 \geq |\lambda| > 1 > \|y\|$. Now, we will distinguish several cases:

1. $|\lambda| \leq \|y\| = 1$ and $|\gamma| \leq \|z\| = 1$. Then, $\frac{\lambda + \gamma}{2} \leq \frac{|\lambda| + |\gamma|}{2} \leq \frac{\|y\| + \|z\|}{2} = 1$, thus it must be $\frac{\lambda + \gamma}{2} \leq \frac{\|y\| + \|z\|}{2} = 1$, and hence $y = z$ because $Y$ is rotund.

2. $|\lambda| \leq \|y\| = 1$ and $|\gamma| > 1 > \|z\|$. We will distinguish two cases here:
   (a) $\gamma > 0$. We have that $\gamma^2 + \|z\|^2 = 2\gamma$ and $(\lambda + \gamma)^2 + \|y + z\|^2 = 4(\lambda + \gamma)$. Then, $2\langle y, z \rangle = -\lambda^2 + (4 - 2\gamma)\lambda + (2\gamma - 1)$.
   (b) $\gamma < 0$. We have that $\gamma^2 + \|z\|^2 = -2\gamma$ and $(\lambda + \gamma)^2 + \|y + z\|^2 = -4(\lambda + \gamma)$. Then, $2\langle y, z \rangle = -\lambda^2 - (4 + 2\gamma)\lambda - (2\gamma + 1)$.

3. $|\lambda| > 1 > \|y\|$ and $|\gamma| > 1 > \|z\|$. Then, it must be $\frac{\lambda + \gamma}{2} > 1 > \frac{\|y\| + \|z\|}{2}$.

We will distinguish two cases here:

(a) $\lambda \gamma > 0$. In this case we can assume that $\lambda \gamma > 0$. We have that $\lambda^2 + \|y\|^2 = 2\lambda$, $\gamma^2 + \|z\|^2 = 2\gamma$ and $(\lambda + \gamma)^2 + \|y + z\|^2 = 4(\lambda + \gamma)$. Eventually we obtain that $\lambda \gamma + \langle y, z \rangle = \lambda + \gamma$, that is $|\lambda + \gamma - \lambda \gamma| = \|\lambda \gamma + \langle y, z \rangle\| = \|\lambda \gamma\| = \sqrt{2\lambda^2 - \lambda^2} - \sqrt{2\gamma^2 - \gamma^2}$. After raising to the square and simplifying, we obtain that $\lambda^2 + \gamma^2 \leq 2\lambda \gamma$. If $y \neq z$, then $\|\langle y, z \rangle\| < \|\lambda \gamma\| = 4(\lambda + \gamma)$.

(b) $\lambda \gamma < 0$. In this case we can assume that $\lambda > -\gamma > 0$. We have that $\lambda^2 + \|y\|^2 = 2\lambda$, $\gamma^2 + \|z\|^2 = -2\gamma$ and $(\lambda + \gamma)^2 + \|y + z\|^2 = 4(\lambda + \gamma)$. Eventually we obtain that $\lambda \gamma + \langle y, z \rangle = \lambda + 3\gamma$, that is $|\lambda + 3\gamma - \lambda \gamma| = \|\lambda \gamma + \langle y, z \rangle\| = \sqrt{2\lambda^2 - \lambda^2} - \sqrt{-2\gamma^2 + \gamma^2}$. After raising to the square and simplifying, we obtain that $(\lambda + 3\gamma)^2 \leq 4\lambda \gamma (-1 + \lambda + \gamma) < 0$, which is impossible.

To conclude the proof, we will show that $H$ does not admit rotund dense and maximal subspaces. Let $f : H \to \mathbb{R}$ be a non-continuous linear functional. Fix $1 < \gamma < 2$. The Intermediate Value Theorem assures that there exists $\lambda \in [-1, 1]$ such that $-\lambda^2 + (4 - 2\gamma)\lambda + (2\gamma - 1) = 0$. Since $f|_Y$ is not continuous, there exists $y \in S_Y$ such that $f(y) = -\lambda f(x)$. Let $Z$ be the orthogonal complement of $\mathbb{R}y$ in
Again \( f |_Z \) is not continuous, therefore there must exist \( z \in \sqrt{2\gamma} - \gamma^2 S_Z \) such that \( f (z) = -\gamma f (x) \). Finally, notice that
\[
\| \lambda x + y \| = \| \gamma x + z \| = \left\| \frac{\lambda + \gamma}{2} x + \frac{y + z}{2} \right\| = 1.
\]
\( \square \)

At this point we need to find a different way to construct an infinite dimensional Banach space with few segments so that it admits a rotund dense subspace. This new way will be opened by making use of the \( L^2 \)-norm and complementation.

**Lemma 2.8.** Let \( Y \) and \( Z \) be Banach spaces. Let \( y_1, y_2 \in Y \) and \( z_1, z_2 \in Z \) such that \([y_1 + z_1, y_2 + z_2] \subset S_{Y \oplus Z} \). Then:

1. \( \| y_1 \| = \| y_2 \| \) and if \( y_1 \neq 0 \), then \( \left[ \frac{y_1}{\| y_1 \|}, \frac{y_2}{\| y_2 \|} \right] \subset S_Y \).
2. \( \| z_1 \| = \| z_2 \| \) and if \( z_1 \neq 0 \), then \( \left[ \frac{z_1}{\| z_1 \|}, \frac{z_2}{\| z_2 \|} \right] \subset S_Z \).

**Proof.** Observe that
\[
4 = \| y_1 + y_2 \|^2 + \| z_1 + z_2 \|^2 \\
\leq \| y_1 \|^2 + \| y_2 \|^2 + 2 \| y_1 \| \| y_2 \| + \| z_1 \|^2 + \| z_2 \|^2 + 2 \| z_1 \| \| z_2 \| \\
= 2 + 2 \| y_1 \| \| y_2 \| + \| z_1 \| \| z_2 \| \\
\leq 2 + 2 \sqrt{\| y_1 \|^2 + \| z_1 \|^2} \sqrt{\| y_2 \|^2 + \| z_2 \|^2} \\
= 4.
\]

On the one hand we conclude that \( \| y_1 + y_2 \| = \| y_1 \| + \| y_2 \| \) and \( \| z_1 + z_2 \| = \| z_1 \| + \| z_2 \| \), therefore \( \left[ \frac{y_1}{\| y_1 \|}, \frac{y_2}{\| y_2 \|} \right] \subset S_Y \) if \( y_1, y_2 \neq 0 \) and \( \left[ \frac{z_1}{\| z_1 \|}, \frac{z_2}{\| z_2 \|} \right] \subset S_Z \) if \( z_1, z_2 \neq 0 \). On the other hand we conclude that \( \| y_1 \| \| y_2 \| + \| z_1 \| \| z_2 \| = \sqrt{\| y_1 \|^2 + \| z_1 \|^2} \sqrt{\| y_2 \|^2 + \| z_2 \|^2} \), which means that there exists \( \alpha > 0 \) such that \( \| y_1 \| = \alpha \| y_2 \| \) and \( \| z_1 \| = \alpha \| z_2 \| \). Then, \( 1 = \| y_1 \|^2 + \| z_1 \|^2 = \alpha^2 \left( \| y_2 \|^2 + \| z_2 \|^2 \right) = \alpha^2 \), that is, \( \alpha = 1 \). \( \square \)

**Theorem 2.9.** Let \( Y \) denote the 2-dimensional Banach space from Lemma 2.4. Let \( Z \) be an infinite dimensional rotund Banach space. Then, \( X := Y \oplus Z \) has a rotund, dense, and maximal subspace.

**Proof.** Let \( x, y \in S_Y \) as in Lemma 2.4. Let \( f : X \to \mathbb{R} \) be a non-continuous linear functional such that \( f (x) = 0 \) and \( f (y) = 1 \). We will show that \( \ker (f) \) is rotund. Let \( y_1, y_2 \in Y \) and \( z_1, z_2 \in Z \) such that \([y_1 + z_1, y_2 + z_2] \subset S_{\ker (f)} \). In virtue of Lemma 2.8 and by hypothesis, we have that \( \| y_1 \| = \| y_2 \| \), \( z_1 = z_2 \), and there exist \( \lambda_1, \lambda_2 \in [0, 1] \) such that \( \frac{y_1}{\| y_1 \|} = \lambda_1 y + (1 - \lambda_1) x \) and \( \frac{y_2}{\| y_2 \|} = \lambda_2 y + (1 - \lambda_2) x \). Since \( f (y_1) = -f (z_1) = -f (z_2) = f (y_2) \), we have that \( \lambda_1 = \lambda_2 \). Therefore, \( y_1 = y_2 \). \( \square \)

**Proof of Theorem A.** Let \( V \) be any 2-dimensional subspace of \( X \). Let \( Z \) be a topological complement for \( V \) in \( X \). We know that \( V \) is isomorphic to \( Y \), where \( Y \) denotes the 2-dimensional Banach space from Lemma 2.4. Then, \( X \) is isomorphic to \( Y \oplus Z \). Finally, we apply Theorem 2.9. \( \square \)
3. Banach spaces with dense locally uniformly rotund subspaces

The beginning of this section is to show that the second construction of the previous section does not work in this case.

**Lemma 3.1.** Let $Y$ and $Z$ be Banach spaces. Let $(y_n)_{n \geq 0} \subset Y$ and $(z_n)_{n \geq 0} \subset Z$ such that $(y_n + z_n)_{n \geq 0} \subset S_{Y \oplus Z}$ and $\|(y_0 + z_0) + (y_n + z_n)\|_{n \in \mathbb{N}}$ converges to 2. Then:

1. $\|y_n\|_{n \in \mathbb{N}}$ converges to $\|y_0\|$ and if $y_0 \neq 0$ then $\left(\left\|\frac{y_0}{\|y_0\|} + \frac{y_n}{\|y_0\|}\right\|\right)_{n \in \mathbb{N}}$ converges to 2.
2. $\|z_n\|_{n \in \mathbb{N}}$ converges to $\|z_0\|$ and if $z_0 \neq 0$ then $\left(\left\|\frac{z_0}{\|z_0\|} + \frac{z_n}{\|z_0\|}\right\|\right)_{n \in \mathbb{N}}$ converges to 2.

**Proof.** Observe that for all $n \in \mathbb{N}$ we have that

\[
4 \leftarrow \|y_0 + y_n\|^2 + \|z_0 + z_n\|^2 \\
\leq \|y_0\|^2 + \|y_n\|^2 + 2\|y_0\|\|y_n\| + \|z_0\|^2 + \|z_n\|^2 + 2\|z_0\|\|z_n\| \\
= 2 + 2\left(\|y_0\|\|y_n\| + \|z_0\|\|z_n\|\right) \\
\leq 2 + 2\sqrt{\|y_0\|^2 + \|z_0\|^2} \sqrt{\|y_n\|^2 + \|z_n\|^2} \\
= 4.
\]

On the one hand we conclude that $\left(\left\|\frac{y_0}{\|y_0\|} + \frac{y_n}{\|y_0\|}\right\|\right)_{n \in \mathbb{N}}$ converges to 0 as well as $\left(\left\|\frac{z_0}{\|z_0\|} + \frac{z_n}{\|z_0\|}\right\|\right)_{n \in \mathbb{N}}$. On the other hand we conclude that $\left(\sqrt{\|y_0\|^2 + \|z_0\|^2} \sqrt{\|y_n\|^2 + \|z_n\|^2} - (\|y_0\|\|y_n\| + \|z_0\|\|z_n\|)\right)_{n \in \mathbb{N}}$ converges also to 0, which means that $\left(\|y_n\|\right)_{n \in \mathbb{N}}$ converges to $\|y_0\|$ and $\left(\|z_n\|\right)_{n \in \mathbb{N}}$ converges to $\|z_0\|$. Everything together gives us the final conclusion of the theorem. \(\square\)

**Theorem 3.2.** Let $Y$ be a non-rotund 2-dimensional Banach space. Let $Z$ be any infinite dimensional Banach space. Then, $Y \oplus Z$ does not have a dense and maximal locally uniformly rotund subspace.

**Proof.** Let $f : Y \oplus Z \to \mathbb{R}$ be a non-continuous linear functional. Let $x \neq y \in S_Y$ such that $[x, y] \subset S_Y$. There exists $z \in S_Z$ such that $f(z) = -f(y)$. Note that $f_Z$ cannot be continuous because $Y$ is 2-dimensional. Therefore $f^{-1}(\{-f((\frac{x+y}{2}))\}) \subset Z$ is dense in $Z$, and since $B_Z$ has non-empty interior in $Z$ we deduce that the set $f^{-1}(\{-f((\frac{x+y}{2}))\}) \cap B_Z$ is dense in $B_Z$. Then, there exists a sequence $(z_n)_{n \in \mathbb{N}} \subset f^{-1}(\{-f((\frac{x+y}{2}))\}) \cap B_Z$ converging to $z$. Finally,

\[
\left(\left\|\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right) + \left(\frac{1}{\sqrt{2}}z_n + \frac{1}{\sqrt{2}}z\right)\right\|\right)_{n \in \mathbb{N}}
\]

converges to 2. \(\square\)

Actually, the previous theorem can be improved in a really surprising way.

**Lemma 3.3.** Let $X$ be a normed space and consider $Y$ to be a dense subspace of $X$. Then:

1. $\text{rot}_a(B_Y) \subseteq \text{rot}_a(B_X)$.
2. If $C$ is any non-trivial segment of $S_X$, then $\text{rot}_a(B_Y) \cap C = \emptyset$.

**Proof.**
(1) It is a simple exercise so we spare the details to the reader.
(2) It is a direct consequence of the previous paragraph.

Observe that the second paragraph of the previous lemma was somehow announced in Remark 2.3.

**Theorem 3.4.** Let $Y$ be a non-rotund 2-dimensional Banach space. Let $Z$ be any infinite dimensional Banach space. Then, $Y \oplus_{2} Z$ does not have a dense and locally uniformly rotund subspace.

**Proof.** Let $V$ be any dense subspace of $Y \oplus_{2} Z$. Let $C$ be a non-trivial segment of $Y$. Let $c$ be in the interior of $C$. By density of $V$, there exists $y \in Y$ and $z \in Z$ such that $y + z \in V$ and $y + z$ is closed enough to $c$ so that $y$ is in the interior of $C$. In accordance to Lemma 2.8,

$$\left[ \frac{y + z}{\|y + z\|_2}, \frac{c + z}{\|c + z\|_2} \right] \subseteq S_{Y \oplus_{2} Z}.$$

Finally, by applying the second paragraph of the previous lemma, $V$ cannot be locally uniformly rotund. □

Unfortunately, the first construction of the previous section does not work either.

**Theorem 3.5.** Let $H$ be an infinite dimensional Hilbert space. Let $x \in S_H$ and $Y$ the orthogonal complement of $\mathbb{R}x$. Consider the equivalent norm on $H$ given by

$$\|\lambda x + y\| := \begin{cases} \frac{\lambda^2 + \|y\|^2}{2|\lambda|} & \text{if } \|y\| < |\lambda| \\ \|y\| & \text{if } |\lambda| \leq \|y\| \end{cases}$$

where $\lambda \in \mathbb{R}$ and $y \in Y$. Then, $H$ does not have dense and locally uniformly rotund subspaces under this new equivalent norm.

**Proof.** The reader may realize that any dense subspace of $H$ will intersect of the sets $[-1, 1] x + y$ for $y \in S_Y$, which are segments according to Theorem 2.7. The second paragraph of Lemma 3.3 will finish the proof. □

Nevertheless, of course the “locally uniformly rotund” version of Lemma 2.2 trivially holds.

**Lemma 3.6.** Let $X$ be an infinite dimensional Banach space enjoying the following property: there exist $x \neq y \in S_X$ such that $\text{rot}_{u}(B_X) = S_X \setminus ([x, y] \cup [-x, -y])$. Then, $X$ is non-rotund but has a locally uniformly rotund dense and maximal subspace.

Bearing in mind Lemma 3.6, our objective then is to construct an infinite dimensional Banach space with only two maximal segments in its unit sphere.

**Lemma 3.7.** Let $B$ be the unit ball of a three dimensional normed space in $\mathbb{R}^3$ that is symmetric with respect to the $z$-coordinate. If $x, y, z_1, z_2 \in \mathbb{R}$ are such that $0 < z_1 \leq z_2$, $(x, y, z_1) \in S_B$ and $(x, y, z_2) \in B$, then $(x, y, z_2) \in S_B$.

**Proof.** Since $B$ is symmetric with respect to the $z$-coordinate with have that $(x, y, -z_1) \in S_B$ and therefore $(x, y, 0) \in B$. Now,

$$(x, y, z_1) = \frac{z_1}{z_2} (x, y, z_2) + \left( 1 - \frac{z_1}{z_2} \right) (x, y, 0),$$

which means that $(x, y, z_2), (x, y, 0) \in S_B$. □
Proof of Theorem B. Let $W$ be the infinite dimensional locally uniformly rotund Banach space. We can assume that the two maximal segments of the 3-dimensional unit ball are $[(1, 0, 0), (0, 1, 0)]$ and its opposite. Let $x \neq \pm y \in S_W$ and let $Z$ be a topological complement for $\mathbb{R}x \oplus \mathbb{R}y$ in $W$. Consider the equivalent norm on $W$ given by

$$\|\lambda x + \gamma y + z\| := \|(\lambda, \gamma, \|z\|)\|$$

where $\lambda, \gamma \in \mathbb{R}$ and $z \in Z$ and the norm in the right hand side is the norm of the 3-dimensional space in the hypothesis. Suppose that $\lambda_1, \lambda_2, \gamma_1, \gamma_2 \in \mathbb{R}$ and $z_1, z_2 \in Z$ are such that

$$\|\lambda_1 x + \gamma_1 y + z_1\| = \|\lambda_2 x + \gamma_2 y + z_2\| = \left\| \frac{\lambda_1 + \lambda_2}{2} x + \frac{\gamma_1 + \gamma_2}{2} y + \frac{z_1 + z_2}{2} \right\| = 1.$$

Now, we can apply Lemma 3.7 to

$$\left( \frac{\lambda_1 + \lambda_2}{2}, \frac{\gamma_1 + \gamma_2}{2}, \frac{z_1 + z_2}{2} \right)$$

to conclude that

$$\left( \frac{\lambda_1 + \lambda_2}{2}, \frac{\gamma_1 + \gamma_2}{2}, \frac{z_1 + z_2}{2}, \frac{\|z_1\| + \|z_2\|}{2} \right)$$

is in the unit sphere of the 3-dimensional Banach space. Therefore, $z_1 = z_2 = 0$ and the only two maximal segments of the new unit ball of $W$ are $[x, y]$ and its opposite. Finally, let us show that if $\lambda, \gamma \in \mathbb{R}$ and $z \in Z \setminus \{0\}$ are such that $\|\lambda x + \gamma y + z\| = 1$, then $\lambda x + \gamma y + z$ is a locally uniformly rotund point. Indeed, let $(\lambda_n)_{n \in \mathbb{N}}, (\gamma_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ and $(z_n)_{n \in \mathbb{N}} \subset Z$ such that $\|\lambda_n x + \gamma_n y + z_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\left\| \left( \frac{\lambda + \lambda_n}{2}, \frac{\gamma + \gamma_n}{2}, \frac{z + z_n}{2} \right) \right\| = \left\| \frac{\lambda + \lambda_n}{2} x + \frac{\gamma + \gamma_n}{2} y + \frac{z + z_n}{2} \right\| \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Observe that

$$\left\| \left( \frac{\lambda + \lambda_n}{2}, \frac{\gamma + \gamma_n}{2}, \frac{\|z\| + \|z_n\|}{2} \right) \right\| \leq 1$$

for all $n \in \mathbb{N}$. We can assume without loss of generality that

$$\left( \frac{\lambda + \lambda_n}{2} \right)_{n \in \mathbb{N}}, \left( \frac{\gamma + \gamma_n}{2} \right)_{n \in \mathbb{N}}, \left( \frac{z + z_n}{2} \right)_{n \in \mathbb{N}}, \left( \frac{\|z\| + \|z_n\|}{2} \right)_{n \in \mathbb{N}}$$

are respectively convergent to $\alpha, \beta, u_1, u_2 \in \mathbb{R}$. Then, we have that $\|(\alpha, \beta, u_1)\| = 1$, $\|(\alpha, \beta, u_2)\| \leq 1$, and $0 \leq u_1 \leq u_2$, therefore Lemma 3.7 assures that $(\alpha, \beta, u_1)$ and $(\alpha, \beta, u_2)$ are on a segment in the unit sphere of the 3-dimensional ball, which means that $u_1 = u_2$. Hence $(\|z + z_n\| - (\|z\| + \|z_n\|))$ converges to 0. Now, $(\lambda, \gamma, \|z\|)$ is a locally uniformly rotund point of the 3-dimensional unit ball, therefore $(\lambda_n)_{n \in \mathbb{N}}, (\gamma_n)_{n \in \mathbb{N}}$ and $(\|z_n\|)$ must converge to $\lambda, \gamma$ and $\|z\|$ respectively. Since $Z$ is locally uniformly rotund, we deduce that $(z_n)_{n \in \mathbb{N}}$ converges to $z$. What is remaining is to apply Lemma 3.6.

Next we construct a closed surface in $\mathbb{R}^3$ which contains two obvious symmetric maximal segments and for each fixed $-1 < z < 1$, the projection onto the $x-y$ plane is an ellipse. This example is a possible candidate for the hypothesis of Theorem B. But we do not know how to verify.
Definition 4.1. Let $X$ be a normed space. Let $x \in S_X$. Then:

1. The point $x$ is said to be a Gateaux-smooth point of $B_X$ (or simply a smooth point of $B_X$) if the norm of $X$ is Gateaux-differentiable at $x$, or equivalently, there exists only one $x^* \in S_X^*$ such that $x^*(x) = 1$, or equivalently, if $(x_n^*)_{n \in \mathbb{N}}$ is a sequence of $B_{X^*}$ such that $(x_n^*(x))_{n \in \mathbb{N}}$ converges to 1, then $(x_n^*)_{n \in \mathbb{N}}$ is $\omega^*$-convergent. The set of Gateaux-smooth points of $B_X$ is usually denoted as $\text{smo}(B_X)$.

2. The point $x$ is said to be a Fréchet-smooth point of $B_X$ (or simply a strongly smooth point of $B_X$) if the norm of $X$ is Fréchet-differentiable at $x$, or equivalently, if $(x_n^*)_{n \in \mathbb{N}}$ is a sequence of $B_{X^*}$ such that $(x_n^*(x))_{n \in \mathbb{N}}$ converges to 1, then $(x_n^*)_{n \in \mathbb{N}}$ is convergent. The set of Fréchet-smooth points of $B_X$ is usually denoted as $\text{smo}_2(B_X)$.

A normed space is said to be Gateaux-smooth or Fréchet-smooth if every point on the unit sphere is a Gateaux-smooth point or a Fréchet-smooth point. The beginning of this section is to show that the second construction of the second section does not apply to the smooth case.

Lemma 4.2. Let $Y$ and $Z$ be Banach spaces. Let $y \in Y$, $z \in Z$, $y^* \in Y^*$, and $z^* \in Z^*$ such that $y + z \in S_{Y \oplus Z}$, $y^* + z^* \in S_{Y^* \oplus Z^*}$ and $(y^* + z^*)(y + z) = 1$. Then:

1. $\|y^*\| = \|y\|$ and if $y \neq 0$, then $\frac{y^*}{\|y\|} \left(\frac{y}{\|y\|}\right) = 1$.
2. $\|z^*\| = \|z\|$ and if $z \neq 0$, then $\frac{z^*}{\|z\|} \left(\frac{z}{\|z\|}\right) = 1$.

Proof. Note that

$$
1 = (y^* + z^*)(y + z) \\
= y^*(y) + z^*(z) \\
\leq \|y^*\| \|y\| + \|z^*\| \|z\| \\
\leq \sqrt{\|y^*\|^2 + \|z^*\|^2} \sqrt{\|y\|^2 + \|z\|^2} \\
= 1.
$$

On the one hand we conclude that $y^*(y) = \|y^*\| \|y\|$ and $z^*(z) = \|z^*\| \|z\|$. On the other hand we conclude that $\|y^*\| \|y\| + \|z^*\| \|z\| = \sqrt{\|y^*\|^2 + \|z^*\|^2} \sqrt{\|y\|^2 + \|z\|^2}$, which means that there exists $\alpha > 0$ such that $\|y^*\| = \alpha \|y\|$ and $\|z^*\| = \alpha \|z\|$. Then, $1 = \|y^*\|^2 + \|z^*\|^2 = \alpha^2 \left(\|y\|^2 + \|z\|^2\right) = \alpha^2$, so $\alpha = 1$. □
Theorem 4.3. Let $Y$ and $Z$ be Banach spaces of dimension greater than or equal to 2 such that $Z$ is infinite dimensional and smooth and $Y$ is finite dimensional. If $Y \oplus Z$ has a dense and maximal smooth subspace, then $Y \oplus Z$ is smooth.

Proof. Let $f : Y \oplus Z \to \mathbb{R}$ be a non-continuous linear functional such that $\ker(f)$ is smooth. Assume to the opposite that $Y \oplus Z$ is not smooth. Then, there are $y \in Y$, $z \in Z$, $y_1', y_2' \in Y^*$, and $z_1', z_2' \in Z^*$ such that $y + z \in S_{Y \oplus Z}$, $y_1' + z_1' \neq y_2' + z_2'$ in $S_{Y \oplus Z}$, and $(y_1' + z_1')(y + z) = 1 = (y_2' + z_2')(y + z)$. Observe that $y_1' \neq y_2'$ or $z_1' \neq z_2'$. Since $Z$ is smooth, we must have $y_1' \neq y_2'$. By Lemma 4.2 we have that $\|y_1\| = \|y_2\|$, $y \neq 0$ and $\frac{y_1}{\|y_1\|} = \frac{y_2}{\|y_2\|}$.

Observe now that $f(y) \neq 0$ because $\ker(f)$ is smooth. Also, $Z$ cannot be entirely contained in $\ker(f)$ because $f|_Y$ is continuous. Therefore, we can find $\omega \in Z$ such that $f(\omega) = -f(y)$. Then, $\frac{y_1}{\sqrt{\|y_1\|^2 + \|\omega\|^2}} + \frac{y_2}{\sqrt{\|y_2\|^2 + \|\omega\|^2}} \in \ker(f)$ and it is not a smooth point. Indeed, let $w^* \in Z^*$ such that $\|w^*\| = \|w\|$ and $\frac{w^*}{\|w^*\|} = \frac{w}{\|w\|}$. Then, $\frac{y_1}{\sqrt{\|y_1\|^2 + \|\omega\|^2}} + \frac{y_2}{\sqrt{\|y_2\|^2 + \|\omega\|^2}}$ are two different functionals in $S_{Y \oplus Z}$ attaining their norm at $\frac{y}{\sqrt{\|y_1\|^2 + \|\omega\|^2}} + \frac{w}{\sqrt{\|y_2\|^2 + \|\omega\|^2}}$. 

Theorem 4.3 forces us to use a different construction to reach our objective. We will use a similar idea as in the first construction of the second section.

Lemma 4.4. Let $X$ be an infinite dimensional Banach space. If there exists $y \in S_X$ verifying that $\text{s}(B_X) = S_X \setminus \{y,-y\}$, then $X$ has a dense and maximal smooth subspace.

Proof. Let $f : X \to \mathbb{R}$ be a non-continuous linear functional such that $f(x) = 0$ and $f(y) = 1$. We know that $\ker(f)$ is a dense and maximal subspace of $X$. We will show that $\ker(f)$ is smooth. Observe that

$$S_{\ker(f)} \subseteq S_X \setminus \{y,-y\} \cap \ker(f) = \text{s}(B_X) \cap \ker(f) \subseteq \text{s}(B_{\ker(f)}).$$

Notice that the Frechet version of Lemma 4.4 trivially holds. That is, we have the following.

Lemma 4.5. Let $X$ be an infinite dimensional Banach space. If there exists $y \in S_X$ verifying that $\text{s}(B_X) = S_X \setminus \{y,-y\}$, then $X$ has a dense and maximal Frechet-smooth subspace.

Lemma 4.6. There exists a 2-dimensional Banach space verifying that there exists $y \in S_X$ with $\text{s}(B_X) = S_X \setminus \{y,-y\}$

Proof. The intersection of the two circles

$$\{(x,y) \in \mathbb{R}^2 : (x+1)^2 + y^2 \leq 2\}$$

and

$$\{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 \leq 2\}$$
determines the unit ball of 2-dimensional Banach space whose only two non-smooth points are \((0, 1)\) and \((0, -1)\). This norm can be described as
\[
\|(x, y)\| = |x| + \sqrt{x^2 + y^2},
\]
where \((x, y)\) ∈ \(\mathbb{R}^2\). Indeed, if \(x > 0\) and \(y > 0\), then
\[
\left( \frac{x}{\|(x, y)\|} + 1 \right)^2 + \left( \frac{y}{\|(x, y)\|} \right)^2 = 2,
\]
so \(\|(x, y)\|\) is the positive solution to the quadratic equation
\[
x^2 + 2xy + (x^2 + y^2) = 0.
\]

\[\square\]

**Theorem 4.7.** Let \(X\) be an infinite dimensional smooth Banach space. Then, \(X\) admits an equivalent renorming such that there exists \(y \in S_X\) verifying that \(\text{smo} (B_X) = S_X \setminus \{y, -y\}\).

**Proof.** Let \(y \in S_X\). Let \(H\) be a topological complement of \(\mathbb{R}y\) in \(X\). Consider the equivalent norm on \(X\) given by
\[
\|h + \lambda y\| := \|h\| + \sqrt{\|h\|^2 + \lambda^2}
\]
where \(h \in H\) and \(\lambda \in \mathbb{R}\). We clearly have that \(\text{smo} (B_X) = S_X \setminus \{y, -y\}\). Indeed, if \(h, k \in H\) and \(\lambda, \gamma \in \mathbb{R}\), then the following limit
\[
\lim_{t \to 0} \frac{\|(h + \lambda y) + t (k + \gamma y)\| - \|h + \lambda y\|}{t} = \lim_{t \to 0} \left( \frac{\|h + tk\| - \|h\|}{t} + \frac{\|h + tk\|^2 + (\lambda + t\gamma)^2 - \|h\|^2 - \lambda^2}{t (\|h + tk\|^2 + (\lambda + t\gamma)^2 + \|h\|^2 + \lambda^2)} \right)\]
\[
= \lim_{t \to 0} \left( \frac{\|h + tk\| - \|h\|}{t} + \frac{1}{2\sqrt{\|h\|^2 + \lambda^2}} \left( \frac{\|h + tk\|^2 - \|h\|^2}{t} + 2\lambda\gamma \right) \right)
\]
is not convergent if and only if \(h = 0\) and \(k \neq 0\).

\[\square\]

**Remark 4.8.** In the settings of Theorem 4.7, the convergence in the limit
\[
\lim_{t \to 0} \frac{\|(h + \lambda y) + t (k + \gamma y)\| - \|h + \lambda y\|}{t}
\]
is uniform on \(k + \gamma y \in S_X\) provided that \(H\) is strongly smooth and that \(h \neq 0\).

**Theorem 4.9.** Let \(X\) be an infinite dimensional Frechet-smooth Banach space. Then, \(X\) admits an equivalent renorming such that there exists \(y \in S_X\) verifying that \(\text{smo}_a (B_X) = S_X \setminus \{y, -y\}\).

Theorem C follows from Lemma 4.5 and Theorem 4.9.
5. NON-COMPLETE NORMED SPACES LACKING THE KREIN-MILMAN PROPERTY

It is a well known problem to determine whether the Krein-Milman Property and the Radon-Nykodim Property are equivalent (see, for instance, [9]).

**Definition 5.1** (Krein and Milman, 1940). Let \( X \) be a topological vector space.

1. Let \( M \) be a convex subset of \( X \) and let \( m \in M \). We will say that \( m \) is an extreme point of \( M \) if \( m \) does not belong to the interior of any segment of \( M \). The set of extreme points of \( M \) is usually denoted by \( \text{ext}(M) \).
2. A bounded closed convex subset \( M \) of \( X \) is said to have the Krein-Milman Property if \( M = \text{cl}(\text{co}(\text{ext}(M))) \).
3. The space \( X \) is said to have the Krein-Milman Property if every bounded closed convex subset of \( X \) has the Krein-Milman property.

Notice that the Krein-Milman Property appears as a direct consequence of the famous Krein-Milman Theorem (see [9]).

**Theorem 5.2** (Krein and Milman, 1940). Let \( X \) be a topological vector space. Let \( K \) be a compact convex subset of \( X \). Then, \( K = \text{cl}(\text{co}(\text{ext}(K))) \).

The Radon-Nikodym Property comes from the theory of vector measures, which is not the purpose of this paper. A suitable characterization of the Radon-Nikodym Property in terms of strongly exposed points directly shows that the Radon-Nikodym Property implies the Krein-Milman Property (see [4]). In [10], the author shows an example of a non-complete normed space where both properties are actually not comparable. Once again, we see the deficiencies of the geometrical properties under the lack of completeness.

**Lemma 5.3.** Let \( X \) be an infinite dimensional topological vector space. Consider \( \{e_i : i \in I\} \subset X \) to be a linearly independent infinite set. Let

\[
M_{00} := \left\{ \sum_{n=1}^{\infty} \lambda_n e_i : (\lambda_n)_{n \in \mathbb{N}} \in \mathcal{B}_{c_{00}} \right\}
\]

and

\[
M_0 := \left\{ \sum_{n=1}^{\infty} \lambda_n e_i : (\lambda_n)_{n \in \mathbb{N}} \in \mathcal{B}_{c_0} \text{ and } \sum_{n=1}^{\infty} \lambda_n e_i \text{ converges in } X \right\}
\]

Then, both \( M_{00} \) and \( M_0 \) are free of extreme points.

**Proof.**

1. If \( \sum_{n=1}^{\infty} \lambda_n e_i \in M_{00} \) with \( (\lambda_n)_{n \in \mathbb{N}} \in \mathcal{B}_{c_{00}}, \) then there exists \( m \in \mathbb{N} \) such that \( \lambda_n = 0 \) for \( n > m \). Then,

\[
\sum_{n=1}^{\infty} \lambda_n e_i = \frac{1}{2} \left( \sum_{n=1}^{m} \lambda_n e_i + e_{i_{m+1}} \right) + \frac{1}{2} \left( \sum_{n=1}^{m} \lambda_n e_i - e_{i_{m+1}} \right).
\]

2. If \( \sum_{n=1}^{\infty} \lambda_n e_i \in M_0 \) with \( (\lambda_n)_{n \in \mathbb{N}} \in \mathcal{B}_{c_0}, \) then there exists \( m \in \mathbb{N} \) such that \( |\lambda_m| < 1 \). Then,

\[
\sum_{n=1}^{\infty} \lambda_n e_i = \frac{1 + \lambda_m}{2} \left( \sum_{n=1}^{m-1} \lambda_n e_i + e_{i_{m}} \right) + \frac{1 - \lambda_m}{2} \left( \sum_{n=1}^{m-1} \lambda_n e_i - e_{i_{m}} \right) + \sum_{n=m+1}^{\infty} \lambda_n e_i.
\]

\[\square\]
The previous lemma gives us the hint of where to start.

**Lemma 5.4.** Let $X$ be a topological vector space that admits a biorthogonal system $(e_i,e_i^*)_{i \in I} \subset X \times X^*$. Then for any subsequence $(e_{i_n})$ of $(e_i)$:

1. $M_{00} := \{\sum_{n=1}^{\infty} \lambda_n e_{i_n} : (\lambda_n)_{n \in \mathbb{N}} \in B_{c_0}\}$ is closed in $\text{span} \{e_i : i \in I\}$.
2. $M_0 := \{\sum_{n=1}^{\infty} \lambda_n e_{i_n} : (\lambda_n)_{n \in \mathbb{N}} \in B_{c_0}$ and $\sum_{n=1}^{\infty} \lambda_n e_{i_n}$ converges in $X\}$ is closed in $\{\sum_{n=1}^{\infty} \lambda_n e_{i_n} : (\lambda_n)_{n \in \mathbb{N}} \in c_0$ and $\sum_{n=1}^{\infty} \lambda_n e_{i_n}$ converges in $X\}$.

**Proof.**

1. Let $x$ be in the closure of $M_{00}$ in $\text{span} \{e_i : i \in I\}$. Then, $x = \sum_{n=1}^{m} \lambda_n e_{i_n}$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. Assume that there exists $j \in \{1, \ldots, m\}$ such that $|\lambda_j| > 1$. Let $0 < \varepsilon < |\lambda_j| - 1$. There exists $y \in M_{00}$ such that $y \in (e_{i_j}^*)^{-1}(|\lambda_j| - \varepsilon, |\lambda_j| + \varepsilon)$. Then, $|e_{i_j}^*(y)| > 1$ which is impossible.

2. It is, essentially, the same proof.

Notice the existence of infinite dimensional topological vector spaces that do not admit biorthogonal systems.

**Remark 5.5.** Let $X$ be a vector space of any dimension. Let us endow $X$ with the trivial topology, that is, $\{\emptyset, X\}$. Then, $X^* = \{0\}$ therefore there are no biorthogonal systems.

Notice also that good geometrical properties like uniform convexity loose strength under the lack of completion.

**Remark 5.6.** Let $(e_n)_{n \in \mathbb{N}}$ be the canonical basis of $\ell_2$. Note that $\text{span} \{e_n : n \in \mathbb{N}\}$ is a pre-Hilbert space (and thus uniformly rotund) but does not enjoy the Krein-Milman Property in virtue of Lemma 5.3 and Lemma 5.4.

Directly from Lemma 5.3 and Lemma 5.4 we can deduce the following.

**Proof of Theorem D.** If the cardinality of a Hamel basis of $X$ is countable infinite, then there exists a countable infinite biorthogonal system $(e_i,e_i^*)_{i \in I} \subset X \times X^*$ which verifies that $X = \text{span} \{e_i : i \in I\}$. At this point we apply Lemma 5.3 and Lemma 5.4. □

To conclude this section we will go over the possible existence of biorthogonal systems.

**Theorem 5.7.** Let $X$ be a locally convex Hausdorff topological vector space. Then, $X$ has a maximal biorthogonal system $(e_i,e_i^*)_{i \in I} \subset X \times X^*$. This biorthogonal system verifies that $\bigcap_{i \in I} \ker (e_i^*) \subseteq \text{span} \{e_i : i \in I\}$. In particular, if this maximal biorthogonal system is total, then it is fundamental and therefore a Markushevich basis.

**Proof.** Define

$$\mathcal{L} := \{(e_i,e_i^*)_{i \in I} \subset X \times X^* : e_i^*(e_j) = \delta_{ij} \text{ for all } i, j \in I\}.$$  

We claim that $\mathcal{L}$ is an inductive set with the partial order given by the inclusion. Indeed, if $(e_i,e_i^*)_{i \in I_k \subset K}$ is a chain of $\mathcal{L}$, then $(e_i,e_i^*)_{i \in \bigcup_{k \in K} I_k \subset \bigcup_{k \in K} I_k} \in \mathcal{L}$. By the Zorn’s Lemma, there exists a maximal element of $\mathcal{L}$ that we will denote by $(e_i,e_i^*)_{i \in I}$.  

Assume that there exists \( x \in \bigcap_{i \in I} \ker (e_i^*) \setminus \operatorname{span} \{ e_i : i \in I \} \). Since \( X \) is Hausdorff and locally convex, the Hahn-Banach Separation Theorem allows us to consider \( x^* \in X^* \) such that \( x^* (x) = 1 \) and \( x^* \left( \operatorname{span} \{ e_i : i \in I \} \right) = \{0\} \). Then,

\[
(e_i, e_i^*)_{i \in I} < (e_i, e_i^*)_{i \in I} \cup (x, x^*) \in \mathcal{L}.
\]

This is a contraction. \( \square \)

**Corollary 5.8.** Let \( X \) be a locally convex Hausdorff topological vector space. If \( X \) admits an infinite fundamental biorthogonal system, then \( X \) has a dense subspace lacking the Krein-Milman Property.

Since every separable infinite dimensional Banach space admits a countable infinite M-basis, Theorem E is a direct consequence of Corollary 5.8.

6. A sufficient condition for an infinite dimensional Banach space to have an infinite dimensional, separable quotient

In 1968, Rosenthal proved the following equivalence version of the Separable Quotient Problem (see [13], [14], and [12]).

**Theorem 6.1** (Rosenthal, 1968). Let \( X \) be an infinite dimensional Banach space. The following conditions are equivalent:

1. \( X \) has an infinite dimensional, separable quotient, that is, there exists a closed subspace \( M \) of \( X \) such that \( X/M \) is infinite dimensional and separable.
2. \( X \) an infinite dimensional, separable quasi-complemented subspace, in other words, there exists an infinite dimensional, separable, closed subspace \( Y \) of \( X \) and another closed subspace \( Z \) of \( X \) such that \( Y \cap Z = \{0\} \) and \( \overline{Y + Z} = X \).

Our contribution to the Separable Quotient Problem in this section is to change quasi-complementation for \( cs \)-complementation in the previous result involving convex series. Therefore, it will be very helpful for our purposes to recall following concepts and results (see [8]).

**Remark 6.2** (Jameson, 1972). Let \( X \) be a topological vector space. A convex series is a series of the form \( \sum_{n=1}^{\infty} t_n x_n \) where \( t_n \geq 0 \) for all \( n \in \mathbb{N} \), \( \sum_{n=1}^{\infty} t_n = 1 \), and \( x_n \in X \) for all \( n \in \mathbb{N} \). If \( A \) is a subset of \( X \), then we say that \( \sum_{n=1}^{\infty} t_n a_n \) is a convex series of \( A \) if it is a convex series and \( a_n \in A \) for all \( n \in \mathbb{N} \).

**Definition 6.3** (Jameson, 1972). Let \( X \) be a topological vector space. subset \( A \) of \( X \) is said to be

1. \( cs \)-closed if every convergent convex series of \( A \) has its sum in \( A \);
2. \( cs \)-compact if every convex series of \( A \) converges to some element of \( A \).

**Theorem 6.4** (Jameson, 1972). Let \( X \) be a topological vector space. Let \( A \) be a subset of \( X \). Then:

1. If \( A \) is \( cs \)-compact, then it is \( cs \)-closed.
2. If \( A \) is \( cs \)-closed, then it is convex.
3. If \( A \) is closed and convex, then it is \( cs \)-closed.
4. If \( A = \text{co} (B \cup C) \), being \( B \) \( cs \)-compact and \( C \) \( cs \)-closed, then \( A \) is \( cs \)-closed.
5. If \( A \) is \( cs \)-closed, then \( A \) and its closure have the same interior.

In case \( X \) is normable, we also have that:
(1) If $A$ is complete, convex, and bounded, then it is cs-compact.
(2) If $A = U_X$, then it is cs-closed.

By taking advantage of the previous concepts, we can redefine (and generalize) the concept of barrel.

**Definition 6.5.** Let $X$ be a topological vector space. A subset $M$ of $X$ is said to be a cs-barrel if it is convex, balanced, absorbing, and cs-closed.

**Remark 6.6.** Let $X$ be a normed space. There always exists a cs-barrel in $X$ that is not a barrel. Indeed, according to the seventh paragraph of Theorem 6.4 it is sufficient to consider the open unit ball $U_X$ of $X$.

**Definition 6.7.** Let $X$ be a topological vector space. Let $Y$ and $Z$ be two vector subspaces of $X$. We say that $Y$ and $Z$ are cs-complemented in $X$ if:

1. $Y$ and $Z$ are linearly quasi-complemented in $X$, that is, $Y \cap Z = \{0\}$ and $Y + Z$ is dense in $X$,
2. $Y$ is closed in $X$, and
3. $Z$ has a cs-barrel $L$ that is cs-closed in $Y + Z$.

**Proposition 6.8.** Let $X$ be a topological vector space. Let $Y$ and $Z$ be two vector subspaces of $X$. If $Y$ and $Z$ are quasi-complemented, then $Y$ and $Z$ are cs-complemented. The converse is not true. As a consequence of Theorem 6.1, if $X$ is a Banach space that admits an infinite dimensional separable quotient, then there exists infinite dimensional closed separable subspace $Y$ of $X$ that is cs-complement in $X$.

**Proof.** Let $L$ be any cs-barrel of $Z$. We need to show that $L$ is cs-closed in $Y + Z$, which is in fact an immediate consequence of the fact that $Y$ and $Z$ are closed and $Y \cap Z = \{0\}$. To see that the converse is not true we consider any two topological vector spaces $Y$ and $Z'$ so that $Z'$ admits a proper dense subspace $Z$. Consider the topological vector space $X := Y \times Z'$ with the product topology. Because of the product topology, $Y \times \{0\}$ and $\{0\} \times Z$ are linearly quasi-complemented in $X$ and $Y \times \{0\}$ is closed in $X$. Finally, if $L$ is any cs-barrel of $\{0\} \times Z$, then $L$ is cs-closed in $Y \times \{0\} + \{0\} \times Z$ because $Y \times \{0\}$ and $\{0\} \times Z'$ are closed in $X$, $Y \times \{0\} \cap \{0\} \times Z' = \{(0,0)\}$, and $\{0\} \times Z$ is cs-closed in $\{0\} \times Z'$.

As we can see, quasi-complementation and cs-complementation are not, in general, equivalent concepts. However, we will show that they are extremely related in the class of Banach spaces that admit an infinite dimensional separable quotient.

**Lemma 6.9.** Let $X$ be an infinite dimensional, separable Banach space. Let $(e_n, e_n^*)_{n \in \mathbb{N}} \subset S_X \times X^*$ be a Markushevich basis. The set

$$N := \left\{ \sum_{n=1}^{\infty} t_n e_n : |t_n| \leq \frac{1}{n^2} \text{ for every } n \in \mathbb{N} \right\}$$

is bounded and closed. In particular, it is a bounded barrel in its linear span, and hence it has empty interior in its linear span.

**Proof.** In the first place, we will show that $N$ is closed. Let $(x_i)_{i \in \mathbb{N}} \subset N$ be a convergent sequence to some $x_0 \in X$. For every $i \in \mathbb{N}$ we can write

$$x_i = \sum_{i=1}^{\infty} t_i e_n$$

where \( |t_n| \leq 1/n^2 \) for every \( n \in \mathbb{N} \). For every \( n \in \mathbb{N} \) we will denote \( e_n^*(x_0) \) by \( t_n^0 \).

If we fix \( n \in \mathbb{N} \), then
\[
\lim_{i \to \infty} t_n^i = \lim_{i \to \infty} e_n^*(x_i) = e_n^*(x_0) = t_n^0,
\]
therefore \( |t_n^0| \leq 1/n^2 \) for every \( n \in \mathbb{N} \). Now, if \( x_0 \neq \sum_{n=1}^{\infty} t_n^0 e_n \), then there is \( m \in \mathbb{N} \) such that
\[
t_m^0 = e_m^*(x_0) \neq e_m^* \left( \sum_{n=1}^{\infty} t_n^0 e_n \right) = t_m^0,
\]
which is impossible. In the second place, we will show that \( N \) is bounded. If we consider \( \sum_{n=1}^{\infty} t_n e_n \) with \( |t_n| \leq 1/n^2 \) for every \( n \in \mathbb{N} \), then
\[
\left\| \sum_{n=1}^{\infty} t_n e_n \right\| \leq \sum_{n=1}^{\infty} |t_n| \leq \sum_{n=1}^{\infty} 1/n^2.
\]
In the third and last place, we will show that \( N \) has empty interior in its linear span. For this it suffices to notice that the set
\[
N \cap \text{span} \{ e_n : n \in \mathbb{N} \} = \left\{ y \in \text{span} \{ e_n : n \in \mathbb{N} \} : |e_n^*(y)| \leq 1/n^2 \text{ for every } n \in \mathbb{N} \right\}
\]
has empty interior in \( \text{span} \{ e_n : n \in \mathbb{N} \} \).

**Theorem 6.10.** Let \( X \) be an infinite dimensional Banach space. If there exists infinite dimensional closed separable subspace \( Y \) of \( X \) that is cs-complement in \( X \), then \( X \) admits an infinite dimensional separable quotient.

**Proof.** Let \( (e_n, e_n^*)_{n \in \mathbb{N}} \subset S_Y \times Y^* \) be a Markushevich basis. By Lemma 6.9 the set
\[
N := \left\{ \sum_{n=1}^{\infty} t_n e_n : |t_n| \leq 1/n^2 \text{ for every } n \in \mathbb{N} \right\}
\]
is bounded, absolutely convex, and complete. In accordance to the fourth paragraph of Theorem 6.4, \( N \) is cs-compact. Let \( Z \) be a cs-algebraical quasi-complement for \( Y \) in \( X \) and \( L \) be a cs-barrel in \( Z \) that is cs-closed in \( Y + Z \). By applying the fifth paragraph of Theorem 6.4 we deduce that \( \text{co} (N \cup L) \) is cs-closed in \( Y + Z \), and hence in \( E := \text{span} (N \cup L) \). So, let \( M \) be the closure of \( \text{co} (N \cup L) \) in \( E \). We will show that \( M \) has empty interior in \( E \). In accordance to the sixth paragraph of Theorem 6.4 it suffices to show that \( \text{co} (N \cup L) \) has empty interior in \( E \). Keep in mind that, by Lemma 6.9, the set \( N \) has empty interior in its linear span. Furthermore,
\[
\text{co} (N \cup L) \cap \text{span} (N) = N.
\]
As a consequence, \( \text{co} (N \cup L) \) must have empty interior in \( E \). Finally, \( E \) is a non-barreled dense subspace of \( X \), therefore \( X \) has an infinite dimensional separable quotient in virtue of [15].

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