THE STRONG APPROXIMATION PROPERTY AND THE WEAK BOUNDED APPROXIMATION PROPERTY

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Abstract. We show that the strong approximation property (strong AP) (respectively, strong CAP) and the weak bounded approximation property (respectively, weak BCAP) are equivalent for every Banach space. This gives a negative answer to Oja's conjecture. As a consequence, we show that each of the spaces c_0 and ℓ_1 has a subspace which has the AP but fails to have the strong AP.

1. Introduction and the main results

One of the most important properties in Banach space theory is the *approximation property* (AP) which was systematically investigated by Grothendieck [G]. A Banach space X is said to have the AP if for every compact subset K of X and every $\varepsilon > 0$, there exists a finite rank continuous linear map (operator) S on X such that $\sup_{x\in K} \|Sx - x\| \leq \varepsilon$, briefly, $id_X \in \overline{\mathcal{F}(X)}^{\tau_c}$, where id_X is the identity map on X, $\mathcal{F}(X)$ is the space of all finite rank operators on X and τ_c is the compact-open topology on the space $\mathcal{L}$ of all operators between Banach spaces. If $id_X \in \overline{\{S \in \mathcal{F}(X) : \|S\| \leq \lambda\}}^{\tau_c}$ for $\lambda \geq 1$, then we say that X has the λ-*bounded approximation property* (λ-BAP).

Oja [O1] introduced a stronger form of the AP. A Banach space X is said to have the *strong approximation property* (strong AP) if for every separable reflexive Banach space Y and every $R \in \mathcal{K}(X,Y)$, the space of all compact operators from X into Y, there exists a bounded net (U_α) in $\mathcal{F}(X,Y)$ such that $\|U_\alpha x - Rx\| \longrightarrow 0$ for every $x \in X$, which is equivalent to that $R \in \overline{\{U \in \mathcal{F}(X,Y) : \|U\| \leq \lambda_R\}}^{\tau_c}$ for some $\lambda_R > 0$ because the compact-open topology and the strong operator topology on $\mathcal{L}$ coincide on any bounded subset of $\mathcal{L}$. This notion is actually stronger than the AP (see [O1, Theorem 2.1] or [K2]).

Lima and Oja [LO] introduced and investigated a weaker form of the BAP. For $\lambda \geq 1$, a Banach space X is said to have the *weak λ-bounded approximation property* (weak λ-BAP) if for every Banach space Y and every $R \in \mathcal{W}(X,Y)$, the space of all weakly compact operators from X into Y, we have $id_X \in \overline{\{S \in \mathcal{F}(X) : \|RS\| \leq \lambda\|R\|\}}^{\tau_c}$. The formal implications between these approximation properties are:

$$\lambda\text{-BAP} \Longrightarrow \text{weak } \lambda\text{-BAP} \Longrightarrow \text{strong AP} \Longrightarrow \text{AP}.$$

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A long standing open problem in this direction is whether the BAP and the AP are equivalent for dual spaces. Grothendieck [G] proved that the 1-BAP and the AP are equivalent for every separable dual space or every reflexive Banach space (see, e.g., [C, Theorems 3.6 and 3.7]). Although a general answer to this problem is not known, some partial results were obtained. It was shown in [LO, Theorem 3.6] that the weak 1-BAP and the AP are equivalent for every dual space and it was shown in [O2, Corollary 1] that the λ-BAP and the weak λ-BAP are equivalent for every Banach space with separable dual. In this paper, we show in Corollary 1.2 that the BAP and the strong AP are equivalent for every Banach space whose dual space is separable.

Lima and Oja [LO] conjectured that the weak BAP and the BAP are different properties and Oja [O1, Conjectures 3.5 and 3.7] conjectured that the strong AP and the weak BAP are different properties. We show in Theorem 1.3 that the strong AP and the weak BAP are actually equivalent for every Banach space. This gives a negative answer to Oja's conjecture.

Figiel and Johnson [FJ] first constructed a Banach space X_{FJ} having the AP but failing to have the BAP. Moreover, its dual space is separable, hence it follows from Corollary 1.2 that the Banach space X_{FJ} does fail to have the strong AP. As a consequence, we show that each of the spaces c_0 and ℓ_1 has a subspace which has the AP but fails the strong AP. It also follows that if X has the AP but its dual space fails to have the AP, then $\ell_p(X)$ $(1 \leq p < \infty)$ has a subspace which has the AP but fails to have the strong AP, where $\ell_p(X)$ is the Banach space of all absolutely p-summable sequences in X.

The main results. We proceed on a more general setting to encompass other approximation properties. Let $T \in \mathcal{L}(X)$ and let $\mathcal{A}(X)$ be a convex subset of $\mathcal{L}(X)$. A Banach space X is said to have the *T-$\mathcal{A}$-approximation property* (T-$\mathcal{A}$-AP) if $T \in \overline{\mathcal{A}(X)}^{\tau_c}$. We say that X has the *strong T-$\mathcal{A}$-approximation property* (strong T-$\mathcal{A}$-AP) if for every Banach space Y and every $R \in \mathcal{K}(X, Y)$, there exists a $\lambda_R > 0$ such that

$$RT \in \overline{\{RS : S \in \mathcal{A}(X), \|RS\| \leq \lambda_R\}}^{\tau_c},$$

in particular, we call the property the strong $\mathcal{A}$-AP, strong T-AP, or strong T-CAP, respectively, if $T = id_X$, $\mathcal{A} = \mathcal{F}$, or $\mathcal{A} = \mathcal{K}$, respectively. This notion is motivated from [O1, Proposition 4.6]. The proof of [O1, Proposition 4.6] actually shows that X has the strong T-AP if and only if for every separable reflexive Banach space Y and every $R \in \mathcal{K}(X, Y)$, there exist a $\lambda_R > 0$ such that $RT \in \overline{\{U \in \mathcal{F}(X, Y) : \|U\| \leq \lambda_R\}}^{\tau_c}$. In view of Proposition 2.1, the strong T-$\mathcal{A}$-AP implies the T-$\mathcal{A}$-AP. We now have:

Theorem 1.1. *Suppose that $\mathcal{A}(X)$ is a convex subset of $\mathcal{K}(X)$ and that X^{**} or Y^* has the Radon-Nikodým property. Let $T \in \mathcal{L}(X)$. If X has the strong T-$\mathcal{A}$-AP, then for every $R \in \mathcal{L}(X, Y)$, there exists a $\lambda_R > 0$ such that $T \in \overline{\{S \in \mathcal{A}(X) : \|RS\| \leq \lambda_R\}}^{\tau_c}$.*

We follow the argument in the proof of [O2, Theorem 2] to show Theorem 1.1 in Section 3 after characterizing the strong T-$\mathcal{A}$-AP in Section 2. From Theorem 1.1, we have:

Corollary 1.2. *Suppose that $\mathcal{A}(X)$ is a convex subset of $\mathcal{K}(X)$ and that X^* or X^{**} has the Radon-Nikodým property. Let $T \in \mathcal{L}(X)$. Then X has the strong T-$\mathcal{A}$-AP if and only if $T \in \overline{\{S \in \mathcal{A}(X) : \|S\| \leq \lambda\}}^{\tau_c}$ for some $\lambda > 0$.*

For $\lambda > 0$, a Banach space X is said to have the weak *T-$\mathcal{A}$-λ-bounded approximation property* (weak T-$\mathcal{A}$-λ-BAP) if for every Banach space Y and every $R \in \mathcal{W}(X,Y)$, we have

$$T \in \overline{\{S \in \mathcal{A}(X) : \|RS\| \leq \lambda \|R\|\}}^{\tau_c}.$$

We call the property the weak $\mathcal{A}$-λ-BAP, weak T-λ-BAP, or weak T-λ-BCAP, respectively, if $T = id_X$, $\mathcal{A} = \mathcal{F}$, or $\mathcal{A} = \mathcal{K}$, respectively. We say that X has the weak T-$\mathcal{A}$-BAP if X has the weak T-$\mathcal{A}$-λ-BAP for some $\lambda > 0$. Clearly, the weak T-$\mathcal{A}$-BAP implies the strong T-$\mathcal{A}$-AP. For the converse, we have:

Theorem 1.3. *Suppose that $\mathcal{A}(X)$ is a convex subset of $\mathcal{K}(X)$ and let $T \in \mathcal{L}(X)$. Then X has the weak T-$\mathcal{A}$-BAP if X has the strong T-$\mathcal{A}$-AP.*

Corollary 1.4. *Each of c_0 and ℓ_1 has a subspace which has the AP but fails to have the strong AP.*

Proof. The case c_0 follows from [FJP, Corollary 1.13] and Corollary 1.2. It was shown in [CKZ, Corollary 1.2] that ℓ_1 has a subspace which has the AP but fails to have the weak BAP. Hence the case ℓ_1 follows from Theorem 1.3. $\square$

It was shown in [CKZ, Theorems 1.1 and 3.4] that if X has the AP (resp. CAP) but its dual space fails to have the AP (resp. conjugate CAP), then each of the spaces $c_0(X)$ and $\ell_p(X)$ $(1 \leq p < \infty)$ has a subspace which has the AP (resp. CAP) but fails to have the weak BAP (resp. weak BCAP). From Theorem 1.3, we have:

Corollary 1.5. *If X has the AP but its dual space fails to have the AP, then the space $\ell_p(X)$ $(1 < p < \infty)$ has a subspace which has the AP but fails to have the strong AP.*

Corollary 1.6. *If X has the CAP but $id_{X^*} \notin \overline{\{S^* : S \in \mathcal{K}(X)\}}^{\tau_c}$, then each of the spaces $c_0(X)$ and $\ell_p(X)$ $(1 \leq p < \infty)$ has a subspace which has the CAP but fails to have the strong CAP.*

2. Characterization of the strong T-$\mathcal{A}$-AP

The following result supplies one of the main tools in the proof of Theorem 1.1.

Proposition 2.1. *Let $T \in \mathcal{L}(X)$ and let $\mathcal{A}(X)$ be a convex subset of $\mathcal{L}(X)$. Then the following statements are equivalent.*
(a) *X has the strong T-$\mathcal{A}$-AP.*
(b) *For every separable reflexive Banach space Y and every $R \in \mathcal{K}(X,Y)$, there exists a $\lambda_R > 0$ such that $RT \in \overline{\{RS : S \in \mathcal{A}(X), \|RS\| \leq \lambda_R\}}^{\tau_c}$.*
(c) *For every Banach space Y and every $R \in \mathcal{K}(X,Y)$, there exists a $\lambda_R > 0$ such that $T \in \overline{\{S \in \mathcal{A}(X) : \|RS\| \leq \lambda_R\}}^{\tau_c}$.*

(a)⇒(b) and (c)⇒(a) in Proposition 2.1 are clear. We need a result of Lima, Nygaard and Oja [LNO] to prove that (b)⇒(c).

Lemma 2.2. *If C is a balanced convex and weakly compact subset of the unit ball B_X of a Banach space X, then there exists a linear subspace Z of X, equipped with a different norm which makes it a reflexive Banach space, such that the inclusion map $J : Z \to X$ has $\|J\| = 1$ and $C \subset B_Z \subset B_X$. Moreover, if C is compact, then the reflexive Banach space Z is separable and the map J is compact.*

Proof of Proposition 2.1(b)⇒(c). We use the argument in the proof of [K1, Proposition 3.1]. Let Y be a Banach space and let $R \in \mathcal{K}(X, Y)$. We may assume $R \neq 0$. We should find a $\lambda_R > 0$ such that $T \in \overline{\{S \in \mathcal{A}(X) : \|RS\| \leq \lambda_R\}}^{\tau_c}$. Suppose not. Then for every $m \in \mathbb{N}$, there exists a $g_m \in (\mathcal{L}(X), \tau_c)^*$ such that

$$\mathrm{Re} g_m(T) > \sup\{\mathrm{Re} g_m(S) : S \in \mathcal{A}(X), \|RS\| \leq m\}.$$

According to the well known Grothendieck's representation [G] of $(\mathcal{L}(X), \tau_c)^*$, for each m, there exist sequences $(x_{n,m})_n$ and $(x^*_{n,m})_n$ in X and X^*, respectively, with $\sum_n \|x_{n,m}\|\|x^*_{n,m}\| < \infty$ such that

$$g_m(U) = \sum_n x^*_{n,m}(Ux_{n,m})$$

for $U \in \mathcal{L}(X)$. We may assume that for every n and m, $\|x^*_{n,m}\| \leq 1$, $\|x^*_{n,m}\| \longrightarrow 0$ as $n \to 0$, and $\sum_n \|x_{n,m}\| < \infty$.

Consider the balanced and closed convex hull C of

$$\Big(\bigcup_{m=1}^{\infty} \Big\{\frac{x^*_{n,m}}{m}\Big\}_{n=1}^{\infty}\Big) \bigcup \Big(\frac{R^*}{\|R\|}(B_{Y^*})\Big),$$

which is a compact subset of B_{X^*}. Then by Lemma 2.2 there exists a separable reflexive Banach space Z, which is a linear subspace of X^*, such that $C \subset B_Z$, the inclusion map $J : Z \to X^*$ is compact and $\|J\| = 1$. By (b) there exists a $\lambda_J > 0$ such that

$$J^* i_X T \in \overline{\{J^* i_X S : S \in \mathcal{A}(X), \|J^* i_X S\| \leq \lambda_J\}}^{\tau_c},$$

where $i_X : X \to X^{**}$ is the natural isometry.

Now choose an $N \in \mathbb{N}$ such that $N \geq \lambda_J \|R\|$. Since $h_N = \sum_n i_Z(x^*_{n,N})(\,\cdot\, x_{n,N}) \in (\mathcal{L}(X, Z^*), \tau_c)^*$, we have

$$\begin{aligned}
\mathrm{Re} g_N(T) &= \mathrm{Re} \sum_n (i_X T x_{n,N}) J(x^*_{n,N}) \\
&= \mathrm{Re} \sum_n (J^* i_X T x_{n,N})(x^*_{n,N}) \\
&= \mathrm{Re} h_N(J^* i_X T) \\
&\leq \sup\{\mathrm{Re} h_N(J^* i_X S) : S \in \mathcal{A}(X), \|J^* i_X S\| \leq \lambda_J\} \\
&= \sup\Big\{\mathrm{Re} \sum_n (J^* i_X S x_{n,N}) x^*_{n,N} : S \in \mathcal{A}(X), \|J^* i_X S\| \leq \lambda_J\Big\} \\
&= \sup\Big\{\mathrm{Re} \sum_n (i_X S x_{n,N}) x^*_{n,N} : S \in \mathcal{A}(X), \|J^* i_X S\| \leq \lambda_J\Big\} \\
&= \sup\{\mathrm{Re} g_N(S) : S \in \mathcal{A}(X), \|J^* i_X S\| \leq \lambda_J\}.
\end{aligned}$$

If $S \in \mathcal{A}(X)$ with $\|J^* i_X S\| \leq \lambda_J$, since $\|J^* i_X S\| = \|S^* J\|$, we have

$$\begin{aligned}
\|RS\| &= \|S^* R^*\| \\
&= \|R\| \sup_{y^* \in B_{Y^*}} \Big\| S^* J\Big(\frac{R^*}{\|R\|}(y^*)\Big)\Big\| \\
&\leq \|R\| \|S^* J\| \leq \lambda_J \|R\|.
\end{aligned}$$

Hence we have

$$\mathrm{Re} g_N(T) \leq \sup\{\mathrm{Re} g_N(S) : S \in \mathcal{A}(X), \|RS\| \leq \lambda_J \|R\|\},$$

which is a contradiction. □

Using a proof similar to that of Proposition 2.1, we obtain the following result which is the main tool in the proof of Theorem 1.3.

Proposition 2.3. *Let $T \in \mathcal{L}(X)$ and let $\mathcal{A}(X)$ be a convex subset of $\mathcal{L}(X)$. Then the following statements are equivalent.*

(a) *For every reflexive Banach space Y and every $R \in \mathcal{W}(X, Y)$, there exists a $\lambda_R > 0$ such that $RT \in \overline{\{RS : S \in \mathcal{A}(X), \|RS\| \leq \lambda_R\}}^{\tau_c}$.*

(b) *For every Banach space Y and every $R \in \mathcal{W}(X, Y)$, there exists a $\lambda_R > 0$ such that $T \in \overline{\{S \in \mathcal{A}(X) : \|RS\| \leq \lambda_R\}}^{\tau_c}$.*

3. A proof of Theorem 1.1

We need a result of Feder and Saphar to prove Theorem 1.1.

Lemma 3.1 (FS, Theorem 1)**.** *Suppose that X^{**} or Y^* has the Radon-Nikodým property. If $f \in (\mathcal{K}(X, Y), \|\cdot\|)^*$, then for every $\varepsilon > 0$, there exist sequences (y^*_n) in Y^* and (x^{**}_n) in X^{**} with $\sum_n \|y^*_n\| \|x^{**}_n\| < \|f\| + \varepsilon$ such that*

$$f(U) = \sum_n x^{**}_n(U^* y^*_n) \quad \textit{for } U \in \mathcal{K}(X, Y).$$

Proof of Theorem 1.1. Let $R \in \mathcal{L}(X,Y)$. We may assume that $R \neq 0$. Suppose that the assertion would be failed. Then for each m, there exists a compact subset K_m of X and $\varepsilon_m > 0$ such that for every $S \in \mathcal{A}(X)$ with $\sup_{x\in K_m}\|Sx - Tx\| \le \varepsilon_m$, $\|RS\| > m$. For each m, let

$$\mathcal{C}_m := \{RS : S \in \mathcal{A}(X), \sup_{x\in K_m}\|Sx - Tx\| \le \varepsilon_m\}.$$

Then we see that for each m, $\mathcal{C}_m$ is nonempty by the assumption, a convex subset of $\mathcal{K}(X,Y)$, and that $\mathcal{C}_m \cap \mathcal{K}(X,Y;m) := \{B \in \mathcal{K}(X,Y) : \|B\| \le m\}$ is empty. Thus for each m, there exists a $f_m \in (\mathcal{K}(X,Y), \|\cdot\|)^*$ with $\|f_m\| = 1$ such that

$$\begin{aligned} m &= \sup\{|f_m(B)| : B \in \mathcal{K}(X,Y;m)\} \\ &= \sup\{\mathrm{Re} f_m(B) : B \in \mathcal{K}(X,Y;m)\} \\ &< \inf\{\mathrm{Re} f_m(A) : A \in \mathcal{C}_m\}. \end{aligned}$$

Since X^{**} or Y^* has the Radon-Nikodým property, it follows from Lemma 3.1 that for each m, there exist sequences $(y^*_{n,m})_n$ in Y^* and $(x^{**}_{n,m})_n$ in X^{**} such that

$$\sum_n \|y^*_{n,m}\|\|x^{**}_{n,m}\| < 2 \quad \text{and} \quad f_m(U) = \sum_n x^{**}_{n,m}(U^* y^*_{n,m}).$$

We may assume that for every m and n, $\|y^*_{n,m}\| \le 1$, $y^*_{n,m} \longrightarrow 0$ as $n \to \infty$, and $\sum_n \|x^{**}_{n,m}\| < 2$. Consider the balanced closed convex hull C, which is a compact subset of B_{X^*}, of

$$\bigcup_{m=1}^{\infty} \Big\{\frac{1}{m}\frac{R^*}{\|R\|}(y^*_{n,m})\Big\}_{n=1}^{\infty}.$$

Then by Lemma 2.2 there exists a separable reflexive Banach space Z, which is a linear subspace of X^*, such that $C \subset B_Z$ and the inclusion map $J : Z \to X^*$ has $\|J\| = 1$. Consider $J^* i_X \in \mathcal{K}(X, Z^*)$. Since X has the strong T-$\mathcal{A}$-AP, by Proposition 2.1 there exists a $\lambda_J > 0$ such that for each m, there exists an $S_m \in \mathcal{A}(X)$ with $\|J^* S^{**}_m\| = \|J^* i_X S_m\| \le \lambda_J$ such that

$$\sup_{x\in K_m}\|S_m x - Tx\| \le \varepsilon_m.$$

Now, for every m, we have

$$\begin{aligned} |f_m(RS_m)| &= \Big|\sum_n x^{**}_{n,m}(S^*_m R^* y^*_{n,m})\Big| \\ &= \Big|\sum_n (S^{**}_m x^{**}_{n,m}) J(R^* y^*_{n,m})\Big| \\ &= \Big|\sum_n (J^* S^{**}_m x^{**}_{n,m})(R^* y^*_{n,m})\Big| \\ &\le \lambda_J \|R\| \sum_n \|x^{**}_{n,m}\| < 2\lambda_J\|R\|. \end{aligned}$$

Thus for every m,

$$\inf\{\mathrm{Re} f_m(A) : A \in \mathcal{C}_m\} \le \inf\{|f_m(A)| : A \in \mathcal{C}_m\} \le |f_m(RS_m)| < 2\lambda_J\|R\|,$$

which is a contradiction. □

In view of Proposition 2.1, from Theorem 1.1, we have:

Corollary 3.2. *Suppose that $\mathcal{A}(X)$ is a convex subset of $\mathcal{K}(X)$ and let $T \in \mathcal{L}(X)$. Then X has the strong T-$\mathcal{A}$-AP if and only if for every Banach space Y whose dual space has the Radon-Nikodým property and every $R \in \mathcal{L}(X,Y)$, there exists a $\lambda_R > 0$ such that $T \in \overline{\{S \in \mathcal{A}(X) : \|RS\| \le \lambda_R\}}^{\tau_c}$.*

From Corollary 3.2, we have:

Corollary 3.3. *If $\mathcal{A}(X)$ is a convex subset of $\mathcal{K}(X)$, then the statements in Proposition 2.1 are equivalent to the ones in Proposition 2.3.*

4. A proof of Theorem 1.3

We need the following lemma, which is due to [K1, Proposition 3.1], to prove Theorem 1.3.

Lemma 4.1. *Let $T \in \mathcal{L}(X)$ and let $\mathcal{A}(X)$ be a convex subset of $\mathcal{L}(X)$. Then X has the weak T-$\mathcal{A}$-BAP if and only if there exists a $\lambda > 0$ such that for every reflexive Banach space Y and every $R \in \mathcal{W}(X,Y)$, we have $RT \in \overline{\{RS : S \in \mathcal{A}(X), \|RS\| \le \lambda\|R\|\}}^{\tau_c}$.*

Proof of Theorem 1.3. Suppose that X would fail to have the weak T-$\mathcal{A}$-BAP. Then by Lemma 4.1, for every $m \in \mathbb{N}$, there exist a Banach space Y_m and $R_m \in \mathcal{W}(X, Y_m)$ such that

$$R_m T \notin \overline{\{R_m S : S \in \mathcal{A}(X), \|R_m S\| \le m^2 \|R_m\|\}}^{\tau_c}.$$

We may assume $\|R_m\| = 1$ for all m. Now, we define the map $R : X \to (\sum_m \oplus Y_m)_{c_0}$ by

$$Rx = \Big(\frac{R_m}{m}(x)\Big)_{m=1}^{\infty}.$$

Then the map is well defined and linear, and it is easily seen that $R \in \mathcal{W}(X, (\sum_m \oplus Y_m)_{c_0})$. One may use a result of Grothendieck (see, e.g., [D, XIII, Lemma 2]) to show it. Thus by the assumption and Corollary 3.3 there exists a $\lambda_R > 0$ such that

$$T \in \overline{\{S \in \mathcal{A}(X) : \|RS\| \le \lambda_R\}}^{\tau_c}.$$

Hence for every m, we have

$$R_m T \in \overline{\{R_m S : S \in \mathcal{A}(X), \|RS\| \le \lambda_R\}}^{\tau_c} \subset \overline{\{R_m S : S \in \mathcal{A}(X), \|R_m S\| \le m\lambda_R\}}^{\tau_c},$$

which is a contradiction. □

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