STABILITY OF BALL PROXIMALITY

PEI-KEE LIN, WEN ZHANG, AND BENTUO ZHENG

Abstract. In this paper, we show that if $E$ is an order continuous Köthe function space and $Y$ is a separable subspace of $X$, then $E(Y)$ is ball proximinal in $E(X)$ if and only if $Y$ is ball proximinal in $X$. As a consequence, $E(Y)$ is proximinal in $E(X)$ if and only if $Y$ is proximinal in $X$. This solves an open problem of Bandyopadhyay, Lin and Rao. It is also shown that if $E$ is a Banach lattice with a 1-unconditional basis and for each $n$, $Y_n$ is a subspace of $X_n$, then $(\oplus Y_n)_E$ is ball proximinal in $(\oplus X_n)_E$ if and only if each $Y_n$ is ball proximinal in $X_n$.

1. Introduction

For any real Banach space $X$, we denote by $B_X$ and $S_X$, respectively, the closed unit ball and unit sphere of $X$. Let $C$ be a nonempty closed convex subset of $X$. For any $x \in X$ and $\delta > 0$, $P_C(x)$ and $P_C(x, \delta)$ denote the sets

$$P_C(x) = \{z \in C : \|x - z\| = d(x, C)\},$$

$$P_C(x, \delta) = \{y \in C : \|x - y\| < d(x, C) + \delta\}.$$

$C$ is said to be proximinal if, for every $x$ in $X$, the set $P_C(x) \neq \emptyset$. A proximinal subset $C$ is said to be strongly proximinal if for any $\varepsilon > 0$ and any $x \in X$, there exists $\delta > 0$, such that $P_C(x, \delta) \subset P_C(x) + \varepsilon B_X$. Let $Y$ be a subspace of $X$. Here subspaces of $X$ always mean closed vector subspaces. $Y$ is said to be ball proximinal (respectively, strongly ball proximinal) if the closed unit ball $B_Y$ is proximinal (respectively, strongly proximinal) [1]. It is easy to see that if $Y$ is a (strongly) ball proximinal subspace of $X$, then $Y$ is a (strongly) proximinal subspace of $X$.

In [10], Saidi showed for any nonreflexive space $Y$ there is a Banach space $X$ such that $Y$ is isometrically isomorphic to a subspace $Z$ of $X$ such that $Z$ is proximinal in $X$; but not ball proximinal in $X$. Bandyopadhyay, Lin, and Rao [1, Example 3.3] showed that the space $Z$ is strongly proximinal in $X$.

Let $(\Omega, \Sigma, \mu)$ be a measure space. A Köthe function space $E$ is a Banach space of measurable real-valued functions that satisfies the following conditions:

1. For any finite measurable set $A$, $\chi_A \in E$.
2. If $g \in E$ and $f$ is a measurable function such that $|f(\omega)| \leq |g(\omega)|$ for almost all $\omega$, then $f \in E$ and $\|f\|_E \leq \|g\|_E$.

$E$ is said to be strictly monotone if for any $0 < f < g$, $\|f\|_E < \|g\|_E$. $E$ is said to be uniformly monotone if for any $\epsilon$, there is $\delta > 0$ such that if $\|f\|_E = 1$ and $\|g\|_E > \delta$, then $\|f + g\| > 1 + \epsilon$. It is easy to see that for any $1 \leq p < \infty$, $L_p$ is uniformly

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monotone. $E$ is said to be order continuous, if for any decreasing sequence $(f_n)$ in $E$ that converges to 0 for almost all $\omega \in \Omega$,
\[
\lim_{n \to \infty} \|f\|_E = 0.
\]

Let $X$ be a Banach space. The Köthe-Bochner function space $E(X)$ is the collection of $X$-valued measurable functions $F$ such that $\|F(\cdot)\|_X \in E$. The norm of $F$ is defined by
\[
\|F\|_{E(X)} = \left\|\|F(\cdot)\|_X\right\|_E.
\]

Several papers [5, 8, 9] have been devoted to studying the following problem:

**Question 1.** Let $E$ be a Köthe function space and $Y$ a subspace of a Banach space $X$. Is $E(Y)$ proximinal in $E(X)$ if $Y$ is proximinal in $X$?

Mendoza proved the following:

1. Let $X$ be a Banach space, let $Y$ be a closed separable subspace of $X$, and let $1 \leq p \leq \infty$. $L_p(\mu, Y)$ is proximinal in $L_p(\mu, X)$ if and only if $Y$ is proximinal in $X$.

2. There exists a Banach space such that $X$ has a proximinal subspace $Y$ such that $L_p([0, 1], Y)$ is not proximinal in $L_p([0, 1], X)$ for any $1 \leq p \leq \infty$.

Bandyopadhyay, Lin, and Rao [1] showed if $(X_n)$ and $(Y_n)$ are two sequences of Banach spaces such that $Y_n$ is a subspace of $X_n$ that is ball proximinal in $X_n$ for each $n$, then $(\oplus Y_n)c_0$ is ball proximinal in $(\oplus X_n)c_0$. They asked

**Question 2.** Let $E$ be a Köthe function space and $X$ a Banach space. Is $E(Y)$ ball proximinal in $E(X)$ if $Y$ is ball proximinal in $X$?

It is easy to check that the subspace $Y_0$ in [8, Example 3.1] is ball proximinal in $X_0$. Therefore, the answer to the above question is negative in general. A sequence $(e_i)$ in a Banach space $X$ is said to be a Schauder basis if every element $x$ in $X$ has a unique representation $x = \sum a_i e_i$. A Schauder basis $(e_i)$ is said to be 1-unconditional if for all $n \in \mathbb{N}$, all finite sequences $(a_i)_{i=1}^n$ of real numbers and all sequences $(\delta_i)_{i=1}^n \subset \{-1, 1\}^n$,
\[
\|\sum_{i=1}^n \delta_i a_i e_i\| = \|\sum_{i=1}^n a_i e_i\|.
\]

In Section 2, we study the above question and we prove the following:

1. Let $E$ be a Banach space with a 1-unconditional basis and for each $n$, $Y_n$ a subspace of $X_n$. $(\oplus Y_n)_E$ is ball proximinal in $(\oplus X_n)_E$ if $Y_n$ is ball proximinal in $X_n$.

2. Let $E$ be an order continuous Köthe function space and $Y$ a separable subspace of $X$. $E(Y)$ is ball proximinal in $E(X)$ if $Y$ is ball proximinal in $X$.

It follows easily from (2) that if $E$ is an order continuous Köthe function space and $Y$ is a separable subspace of $X$, then $E(Y)$ is ball proximinal in $E(X)$ if and only if $Y$ is ball proximinal in $X$.

It worth to mention that all the results and proofs in this paper are also true for complex Banach spaces by modifying corresponding notations. For example, if we let $X$ be a complex Banach space and let Köthe function spaces be complex valued, we can define $E^+ = \{f \in E : f(\omega) \geq 0, \text{ for almost all } \omega \in \Omega\}$. Then we define $E$ to be strictly monotone if for any $|f| < |g|$, $\|f\| < \|g\|$. 
2. BALL PROXIMALITY

Let $E$ be a Banach space with a 1-unconditional basis $(e_n : n \in \mathbb{N})$. Classical sequence spaces $\ell_p$ (1 ≤ $p$ < ∞) or $c_0$ and separable Orlicz sequence spaces have 1-unconditional bases. Let $(X_n)$ be a sequence of Banach spaces. The space $(\sum \oplus X_n)_E$ is the collection of elements $(x_n)$ such that $x_n \in X_n$ and $\sum_{n=1}^{\infty} \|x_n\| e_n \in E$. The norm of $(x_n)$ is defined by

$$\| (x_n) \| (\sum \oplus X_n)_E = \left\| \sum_{n=1}^{\infty} \|x_n\| e_n \right\|_E.$$ 

With this norm, $(\sum \oplus X_n)_E$ becomes a Banach space.

**Lemma 3.** Let $E$ be a Banach space with a 1-unconditional basis $(e_n)$ and $(X_n)$ be a sequence of Banach spaces. Let $(Y_n)$ be a sequence of Banach spaces such that $Y_n$ is a ball proximinal subspace of $X_n$ for each $n$. For each $a = (a_n) = \sum_{n=1}^{\infty} a_n e_n$ in $E^+$ (the set of elements in $E$ with all coordinates non-negative), let

$$A_a = \left\{ (y_n) \in \left( \sum \oplus Y_n \right)_E : \|y_n\| X_n \leq a_n \text{ for all } n \right\}.$$ 

Then $A_a$ is proximinal in $(\sum \oplus X_n)_E$.

**Proof.** Let $(x_n)$ be any element in $(\sum \oplus X_n)_E$. Since $Y_n$ is ball proximinal in $X_n$, there is $z_n \in a_n B_{Y_n}$ such that

$$\|z_n - x_n\| X_n = d_{X_n}(x_n, a_n B_{Y_n}).$$

Then $(z_n) \in A_a$ and for any $(y_n) \in A_a$,

$$\left\| \left( \left\| x_n - z_n \right\| X_n \right) \right\|_E \leq \left\| \left( \left\| x_n - y_n \right\| X_n \right) \right\|_E.$$ 

The proof is complete. 

**Theorem 4.** Let $E$ be a Banach space with a 1-unconditional basis $(e_n)$ and $(X_n)$ a sequence of Banach spaces. If $(Y_n)$ is a sequence of Banach spaces such that $Y_n$ is ball proximinal in $X_n$ for each $n$, then $(\sum \oplus Y_n)_E$ is ball proximinal in $(\sum \oplus X_n)_E$.

**Proof.** Suppose that for each $n$, $Y_n$ is a ball proximinal subspace of $X_n$. Let $x = (x_n)$ be an element in $(\sum \oplus X_n)_E$ and let

$$D_x = \left\{ (a_n) \in B^+_E : a_n \leq 2\|x_n\| X_n \text{ for all } n \right\}.$$ 

Define a function $H$ from $D_x$ to $\mathbb{R}$ by

$$H(a) = d_{(\sum \oplus X_n)_E}(x, A_a) \text{ for any } a \in D_x.$$ 

Then $H$ is a continuous convex function. So $H$ is lower semi-continuous from $D_x$ with weak topology to $\mathbb{R}$. By a Grothendieck’s Theorem [2, Lemma 2 page 227], $D_x$ is a weakly compact convex subset of $E$. So $H$ attains its minimum. By Lemma 3, there is $(y_n) \in \cup a \in D_x A_a$ such that for any $(y'_n) \in \cup a \in D_x A_a$ we have

$$\left\| \left( \left\| y_n - x_n \right\| X_n \right) \right\|_E \leq \left\| \left( \left\| y'_n - x_n \right\| X_n \right) \right\|_E.$$ 

We claim that $(y_n)$ is a best approximation of $(x_n)$ from $B_{(\sum \oplus Y_n)_E}$. Let $(z_n)$ be any element in $B_{(\sum \oplus Y_n)_E}$ and let $(z'_n)$ be the element defined by

$$z'_n = \begin{cases} z_n & \text{if } \|z_n\| X_n \leq 2\|x_n\| X_n, \\ 0 & \text{if } \|z_n\| X_n > 2\|x_n\| X_n. \end{cases}$$
Then \((z_n') \in A_n\) for some \(a \in D_X\). If \(\|z_n\|_{X_n} > 2\|x_n\|_{X_n}\), then
\[
\|z_n' - x_n\|_{X_n} = \|x_n\|_{X_n} \leq \|z_n\|_{X_n} - \|x_n\|_{X_n} \leq \|z_n - x_n\|_{X_n}.
\]
So we have
\[
\|(x_n) - (z_n)\|_{(\sum \oplus X_n)_E} \geq \|(x_n) - (z_n')\|_{(\sum \oplus X_n)_E} \geq \|(x_n) - (y_n)\|_{(\sum \oplus X_n)_E}.
\]
The proof is complete. \(\square\)

**Theorem 5.** Let \(E\) be an order continuous Köthe function space and \(Y\) a separable ball proximinal subspace of a Banach space \(X\). Then \(E(Y)\) is a ball proximinal subspace of \(E(X)\).

**Proof.** Suppose that \(E\) is an order continuous Köthe function space. It is known that [7, Theorem 5.3.14] for any \(g \in E^+\), the set
\[
\{0 \leq f \leq g : f \in E\}
\]
is weakly compact. By the proof of Theorem 4, we need only to show that for any \(F \in E(X)\) and any \(g \in E^+\), there is a measurable \(X\)-valued function \(G\) such that \(G(t) \in P_{g(t)B_Y}(F(t))\).

Let \(\{y_n : n \in \mathbb{N}\}\) be a dense subset of \(B_Y\) and \(g \in E^+\). For each \(n\), let
\[
G_n = \begin{cases} 
0 & \text{if } g(t) = 0 \\
\{(t, y) \in \Omega \times Y_B : \|F(t)/g(t) - y_n\|_{X} \geq \|F(t)/g(t) - y\|_{X} \} & \text{if } g(t) \neq 0.
\end{cases}
\]
Since \(F\) is a measurable function, \(G_n\) is measurable. Let \(G = \bigcap_{n=1}^{\infty} G_n\). Then \(G\) is measurable such that for every \(t \in \Omega\) with \(g(t) \neq 0,
\[
\{y \in B_Y : (t, y) \in G\} = P_{B_Y}(F(t)/g(t)).
\]
By the von Neumann measurable selection theorem, there is a measurable function \(H\) such that if \(g(t) \neq 0\), then \(H(t) \in P_{B_Y}(F(t)/\|g(t)\|)\). Let \(G = g \cdot H\). Then \(G\) is a measurable function such that \(G(t)\) is a best approximate of \(F(t)\) from \(g(t)B_Y\).

The proof is complete. \(\square\)

**Remark 6.** The above two theorems give affirmative answers to the question of Bandypadhyay, Lin, and Rao [1, page 252 and 263].

By Theorem 4 and Theorem 5, we have the following theorems.

**Theorem 7.** Let \(E\) be a strictly monotone Köthe function space and \(Y\) a ball proximinal subspace of a Banach space. If either \(E\) is a Banach space with a 1-unconditional basis or \(Y\) is separable, then for any \(F \in E(X)\) and \(G \in E(Y)\), \(G\) is a best approximation of \(F\) for \(B(E(Y))\) implies that for almost all \(\omega \in \Omega\), \(G(\omega)\) is a best approximation of \(F(\omega)\) from \(\|G(\omega)\|_{B_{Y}}\).

**Proof.** Assume that either \(E\) is a Banach lattice with a 1-unconditional basis or \(Y\) is a separable ball proximinal subspace of \(X\). By the proofs of Theorem 4 and Theorem 5, for any \(g \in B_{E(Y)}^+\), there is \(G \in B_{E(Y)}\) such that \(\|G\|_{X} = g\) and for almost all \(\omega \in \Omega\), \(G(\omega)\) is a best approximation of \(F(\omega)\) from \(g(\omega)B_Y\). If \(H\) is an element in \(B_{E(Y)}\) such that \(\|H(\cdot)\|_{X} = g\). Then \(\|(F - G)(\omega)\|_{X} \leq \|(F - H)(\omega)\|_{X}\).

Since \(E\) is strictly monotone, if \(G \neq H\), then we have \(\|F - G\|_{E(X)} < \|F - H\|_{E(X)}\).

The proof is complete. \(\square\)

**Theorem 8.** Let \(E\) be a Banach space with a 1-unconditional basis \((e_n)\) and \((X_n)\) a sequence of Banach spaces. If \((Y_n)\) is a sequence of Banach spaces such that \(Y_n\) is proximinal in \(X_n\) for each \(n\), then \((\sum \oplus Y_n)_{E}\) is proximinal in \((\sum \oplus X_n)_{E}\).
Theorem 9. Let $E$ be an order continuous Köthe function space and $Y$ a separable proximinal subspace of a Banach space $X$. Then $E(Y)$ is a proximinal subspace of $E(X)$.

3. STRONGLY BALL PROXIMALITY

Let $E$ be a Banach space with 1-unconditional basis $(e_n)$. $(e_n)$ is said to be uniformly monotone if for all $\epsilon > 0$, there exists a $\delta > 0$ so that $\| \sum (|a_n| + |b_n|) e_n \| > 1 + \epsilon$ whenever $\| \sum a_n e_n \| = 1$ and $\| \sum b_n e_n \| > \delta$. It is easy to verify that the unit vector basis of $\ell_p (1 \leq p < \infty)$ is 1-unconditional and uniformly monotone.

Theorem 10. Let $E$ be a Banach space with a uniformly monotone 1-unconditional basis $(e_n)$, $(X_k)$ and $(Y_k)$ be sequences of Banach spaces such that $Y_k$ is strongly ball proximinal in $X_k$ for all $k$. Then $(\sum \oplus Y_k)_E$ is strongly ball proximinal in $(\sum \oplus X_k)_E$.

Proof. For each $m$, let $P_m$ be the projection on $E$ defined by

$$P_m \left( \sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=1}^{m} a_k e_k.$$ 

Suppose that the theorem is not true. Then there are an element $x = (x_k)$ in $(\sum \oplus X_k)_E$, $\epsilon > 0$ and a sequence $\{y_n = (y_{k,n})\}$ from $B((\sum \oplus Y_k)_E)$ such that

$$\| x - y_n \|_{(\sum \oplus X_k)_E} \leq d(x, B((\sum \oplus Y_k)_E)) + \frac{1}{n},$$

but

$$d(y_n, P_B((\sum \oplus Y_k)_E)(x)) \geq \epsilon$$

for all $n$. Without loss of generality, we assume that

$$d(x, B) = 1.$$

We claim that for any $\epsilon > 0$, there are $m$ and $N$ such that

$$\| (I - P_m)(\|y_{k,n}\|_{X_k}) \| \leq \epsilon$$

for all $n > N$.

Suppose that it is not true. Then there is $\epsilon > 0$ for any $N$ and $m$, there is $n > N$ such that

$$\| (I - P_m)(\|y_{k,n}\|_{X_k}) \| > \epsilon.$$ 

By passing to a subsequences of $(y_n)$, we may assume that for any $m$ there is $N_m$ such that

$$\| (I - P_m)(\|y_{k,n}\|_{X_k}) \| > \epsilon$$

for all $n > N_m$. Since $E$ is uniformly monotone, there is $\delta > 0$ such that for any $y, z \in E^+$, with $\frac{1}{2} \leq \|y\|_E \leq 1$ and $\|z\|_E > \epsilon$, we have

$$\| y + z \|_E \geq \|y\|_E + 4\delta.$$
Select $m > \frac{1}{4}$ such that $\| (I - P_m)x \|_{\sum \oplus X_n} < \delta$. If $n > N_m$, then
\[
\left\| x - P_m(y_n) \right\|_{E} \leq \left\| P_m(x - y_n) \right\|_{E} + \left\| (I - P_m)x \right\|_{E} \leq 4\delta + \delta
\]
\[
\leq \| x - y_n \|_{E} + \delta - 3\delta
\]
\[
\leq 1 + \frac{1}{m} - 2\delta \leq 1 - \delta.
\]
We get a contradiction.

By the claim, \{y_n : n \in \mathbb{N}\} is relatively compact. By passing to a further subsequence, we may assume that \((y_{k,n})\) converges, say it converges to $y = (y_k)$. Then for any $z_n \in Y \cap P_{\sum \oplus X_n} B_{Y_n}(x_n)$, we have
\[
\| x - z_n \|_{E} = 1.
\]

Give \( \epsilon > 0 \). Select $m$ large enough so that
\[
\left\| (I - P_m)x \right\|_{E} < \frac{\epsilon}{4},
\]
\[
\left\| (I - P_m)y \right\|_{E} < \frac{\epsilon}{4}.
\]
Since $Y_k$ is strongly ball proximinal in $X_k$ for $k \leq m$, there is $\delta > 0$ such that for any $k \leq m$, if $w_k \in \| y_k \|_{X_k} B_{Y_k}$ and $d(w_k, \| y_k \|_{X_k} B_{Y_k}) < d(z_k, \| y_k \|_{X_k} Y_k) + \delta$, we have
\[
d(z_k, P_{\| y_k \|_{X_k} B_{Y_k}}(x_k)) < \frac{\epsilon}{4m}.
\]
So if $w = (w_n) \in Y \cap P_{\sum \oplus X_n} B_{Y_n}(x_n)$ such that $\| x - w \| < 1 + \delta$, then
\[
d(w, P_{\sum \oplus X_n} E)(x) < \epsilon.
\]
So $(\sum \oplus Y_n)\cap E$ is strongly ball proximinal in $(\sum \oplus X_n)\cap E$.

\[\square\]

Remark 11. Let \( \| \cdot \| \) be the norm on $c_0$ defined by
\[
\left\| (x_n) \right\| = \left\| (x_n) \right\|_\infty + \left( \sum_{n=1}^{\infty} \frac{1}{2^{2n}} |x_n|^2 \right)^{1/2}.
\]
Then $(c_0, \| \cdot \|)$ is not strongly ball proximinal in itself [6, p. 87]. The assumption “$E$ is uniformly monotone” in the above theorem cannot be removed.

Recall that a subspace $Y$ of a Banach space $X$ is equable if for any $\epsilon > 0$, there are $\delta > 0$ and a map $\psi : Y \to [0, 1]$ such that for every $y \in Y$,
\[
\| y - \psi(y) y \| < \epsilon \quad \text{and} \quad B(0, 1) \cap B(y, 1 + \delta) \subset B(\psi(y)y, 1).
\]
Lalithambigai proved that

(1) If $Y$ is an equable subspace of $X$, then $Y$ is strongly ball proximinal in $X$ [6, Theorem 2.6].

(2) Let $(X_n)$ be a sequence of Banach spaces, and for each $n$, $Y_n$ an equable subspace in $X_n$. Then $(\sum \oplus Y_n)_{c_0}$ is an equable subspace of $(\sum \oplus X_n)_{c_0}$ [6, Theorem 3.2].

We show the similar result still holds if we replace equable subspace by strongly ball proximinal subspace.
Theorem 12. Let \((X_n)\) be a sequence of Banach spaces and for each \(n\), \(Y_n\) a strongly ball proximinal subspace of \(X_n\). Then \((\sum \oplus Y_n)_{c_0}\) is strongly ball proximinal in \((\sum \oplus X_n)_{c_0}\).

Proof. Let \(x = (x_n)\) be any element in \((\sum \oplus X_n)_{c_0}\). For each \(n\), let

\[\alpha_n = d_{X_n}(x_n, B_{X_n}).\]

Then \((\alpha_n) \in c_0\). Without loss of generality, we assume that \(\sup \alpha_n = 1\) and let \((y_n)\) be a best approximation of \(x\) from \((\sum \oplus Y_n)_{c_0}\). Then \(\|y_n\|_{X_n} \leq 2\|x_n\|_{X_n}\) for all \(n\).

Let \(\epsilon\) be any positive real and let \(A = \{n : \alpha_n > \frac{1}{2}\}\). Then \(A\) is a finite set. For each \(n \in A\), there is \(\delta_n > 0\) such that if \(z_n \in B_{Y_n}\) and \(\|z_n - x_n\|_{X_n} < \|y_n - x_n\|_{X_n} + \delta_n\), then we have

\[d(z_n, P_{B_{Y_n}}(x_n)) < \epsilon.\]

Let \(A' = \{n \in A : \alpha_n + \delta > 1\}\) and let \(\beta = \sup\{\alpha_n : n \notin A'\}\). Then \(\beta < 1\) and \(\alpha_n = 1\) for some \(n \in A'\).

Given Theorem 10 and Theorem 12, a natural question is under what conditions \((\sum Y_n)_{\ell_\infty}\) is strongly ball proximinal in \((\sum X_n)_{\ell_\infty}\). It looks difficult to give a general answer to this question. A special case of this problem is to find the right Banach space conditions for a Banach space \(X\) under which the unit ball of \(\ell_\infty(X)\) is strongly proximinal in \(\ell_\infty(X)\). To give an answer to this question, we introduce the following definition.

Let \(C \subset X\) be a closed convex subset. We say that \(C\) is uniformly proximinal in \(X\) if for any \(\epsilon > 0\) and \(\alpha > 0\), there is \(\delta_\epsilon(\alpha) > 0\), s.t. for any \(x \in X\) with \(d(x, C) \leq \alpha\), for any \(y \in C\) with \(\|x - y\| < \alpha + \delta_\epsilon(\alpha)\), we have

\[d(y, P_C(x)) < \epsilon\]

Let \(\delta_\epsilon(\alpha)\) be the supremum of all such \(\delta_\epsilon(\alpha)\). \(X\) is said to be uniformly ball proximinal if \(B_X\) is uniformly proximinal in \(X\). It is easy to see that uniformly convex Banach spaces are uniformly ball proximinal.

Remark 13. If \(X\) is uniformly ball proximinal, then the unit ball of \(X\) is strongly proximinal in \(X\). So by Theorem 10, the unit ball of \(\ell_p(X)\) \((1 \leq p < \infty)\) is strongly proximinal in \(\ell_p(X)\).

Question 3. If \(X\) is uniformly ball proximinal, is \(\ell_\infty(X)\) strongly ball proximinal?

Remark 14. It is not hard to see that if \(l_\infty(X)\) is strongly ball proximinal, then \(X\) is uniformly ball proximinal. In the next theorem, we will show the converse is true with one additional condition.

Let \(X\) be uniformly ball proximinal. \(X\) is said to be regular if \(\forall 0 < \alpha < \beta, \forall \epsilon > 0\),

\[\inf_{\alpha \leq \gamma \leq \beta} \{\delta_\epsilon(\gamma)\} > 0.\]

Theorem 15. Let \(X\) be uniformly ball proximinal. If \(X\) is regular, then \(l_\infty(X)\) is strongly ball proximinal.
Proof. Let \((x_n) \in \ell_\infty(X)\) with \(\|x_n\| = 1 + \beta, \, \beta > 0\). For all \(\varepsilon > 0\) (without loss of generality we assume \(\varepsilon < \min\{\beta, 4\}\)), we define
\[
\delta = \frac{\varepsilon}{8} \inf_{\beta - \varepsilon \leq \gamma \leq \beta} \delta_\varepsilon(\gamma).
\]
Take any \((y_n) \in B_{\ell_\infty(X)}\) with \(\|(x_n) - (y_n)\| < \beta + \delta\). Let
\[
I_1 = \{n \in \mathbb{N} : \|x_n - y_n\| \leq \beta\}
\]
and
\[
I_2 = \{n \in \mathbb{N} : \|x_n - y_n\| > \beta\}.
\]
Now we are going to find \((z_n) \in P_{B_{\ell_\infty(X)}}(x_n)\) so that \(\|(y_n) - (z_n)\| \leq \varepsilon\).

If \(n \in I_1\), we simply define \(z_n = y_n\). If \(n \in I_2\), we have the following two cases.

Case 1. \(\|x_n\| > 1 + \beta - \frac{1}{2} \inf_{\beta - \varepsilon \leq \gamma \leq \beta} \delta_\varepsilon(\gamma)\).

Case 2. \(\|x_n\| \leq 1 + \beta - \frac{1}{2} \inf_{\beta - \varepsilon \leq \gamma \leq \beta} \delta_\varepsilon(\gamma)\).

In Case 1, we have
\[
\|x_n - y_n\| < \beta + \delta = (\|x_n\| - 1) + (1 + \beta - \|x_n\|) + \delta
\]
\[
< d(x_n, B_X) + (\frac{1}{2} + \frac{\varepsilon}{8}) \inf_{\beta - \varepsilon \leq \gamma \leq \beta} \delta_\varepsilon(\gamma)
\]
\[
< d(x_n, B_X) + \inf_{\beta - \varepsilon \leq \gamma \leq \beta} \delta_\varepsilon(\gamma)
\]
\[
\leq d(x_n, B_X) + \delta_\varepsilon(d(x_n, B_X)).
\]
The last inequality holds since \(\delta_\varepsilon(\alpha) \leq \varepsilon\) for any \(\varepsilon, \alpha > 0\) and therefore
\[
\beta - \varepsilon < d(x_n, B_X) \leq \beta.
\]

Hence \(\exists z_n \in P_{B_X}(x_n)\) so that \(\|y_n - z_n\| < \varepsilon\).

In Case 2, let \(z_n = (1 - \frac{\varepsilon}{3})y_n + \frac{\varepsilon}{3} \frac{x_n}{\|x_n\|}\). It is clear that \(z_n \in B_X\). We proceed to estimate \(\|z_n - y_n\|\) and \(\|z_n - x_n\|\).

\[
\|z_n - y_n\| = \left\| \frac{\varepsilon}{3}(y_n - \frac{x_n}{\|x_n\|}) \right\| \leq \frac{2\varepsilon}{3} < \varepsilon
\]

and
\[
\|z_n - x_n\| = \left\| (1 - \frac{\varepsilon}{3}) (x_n - y_n) + \frac{\varepsilon}{3} (x_n - \frac{x_n}{\|x_n\|}) \right\|
\]
\[
\leq \left\| (1 - \frac{\varepsilon}{3}) (x_n - y_n) \right\| + \frac{\varepsilon}{3} (\|x_n\| - 1)
\]
\[
< (1 - \frac{\varepsilon}{3}) (\beta + \delta) + \frac{\varepsilon}{3} \left( \beta - \frac{1}{2} \inf_{\beta - \varepsilon \leq \gamma \leq \beta} \delta_\varepsilon(\gamma) \right)
\]
\[
\leq \beta + \left( (1 - \frac{\varepsilon}{3}) - \frac{\varepsilon}{6} \right) \inf_{\beta - \varepsilon \leq \gamma \leq \beta} \delta_\varepsilon(\gamma)
\]
\[
= \beta - \frac{\varepsilon + \varepsilon^2}{24} \inf_{\beta - \varepsilon \leq \gamma \leq \beta} \delta_\varepsilon(\gamma) < \beta.
\]

From our construction, it is easy to see that \((z_n) \in P_{B_{\ell_\infty(X)}}(x_n)\) and \(\|(y_n) - (z_n)\| \leq \varepsilon\).

The following are some related open problems.
Question 4. If $X$ is uniformly ball proximinal, is $\ell_p(X)$ ($1 \leq p < \infty$) uniformly ball proximinal?

Question 5. If $Y$ is a subspace of $X$ and $Y$ is uniformly proximinal in $X$, is $\ell_p(Y)$ ($1 \leq p < \infty$) uniformly ball proximinal in $\ell_p(X)$?

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REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MEMPHIS, MEMPHIS TN, 38152
E-mail address: pklin@memphis.edu

DEPARTMENT OF MATHEMATICS, XIAMEN UNIVERSITY
E-mail address: wenzhang@xmu.edu.cn

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MEMPHIS, MEMPHIS TN, 38152
E-mail address: bzheng@memphis.edu