Math 7351 1. Metric spaces Spring 2005

Many concepts from real analysis can be extended to other spaces if we have a *metric* on the space. A *metric space* is a set X with a 'distance' or 'metric' function $d\colon X \times X \to \mathbb{R}$ such that

D1. $d(x, y) \geq 0$.

D2. $d(x, y) = 0$ iff $x = y$

D3. $d(x, y) = d(y, x)$ (Symmetry).

D4. $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle inequality).

Examples

1. $X = \mathbb{R}$ with $d(x, y) = |x - y|$. This is the usual distance in the reals.
2. More generally, if $(X, \|\cdot\|)$ is a normed space then $d(x, y) = \|x - y\|$ defines a metric. The L^p spaces are examples.
3. In $\mathbb{R}^2$, the standard Euclidean distance $d(x, y) = \|x-y\|_2 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ and the 'Taxicab' metric $d(x, y) = \|x - y\|_1 = |x_1 - y_1| + |x_2 - y_2|$ are metrics (in fact these are just given by the L^2 and L^1 norms respectively on $\mathbb{R}^2$).
4. On any space, $d(x, y) = 0$ if $x = y$ and 1 otherwise gives rise to the *discrete* metric.
5. The 2-adic metric on $\mathbb{Z}$ given by $d(n, m) = 2^{-a}$ if $n - m = 2^a(2k + 1)$ and 0 if $n = m$.
6. FedEx Metric: $X = \mathbb{R}^2$ with $d(x, y) = \|x\|_2 + \|y\|_2$ if $x \neq y$ and 0 if $x = y$.
7. $X = (0, \infty)$ with $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$.

Definition The *(open) ball or radius r about $x \in X$* is the set $B_r(x) = \{y : d(y, x) < r\}$.

Concepts of limits, continuity, and open and closed sets have fairly straightforward generalizations to metric spaces:

A sequence $(x_i)_{i\in\mathbb{N}}$ in a metric space (X, d) *converges to a limit* $L \in X$ iff

$$\forall\varepsilon > 0\colon \exists n_0\colon \forall n \geq n_0\colon d(x_n, L) < \varepsilon \qquad \text{or} \qquad \forall\varepsilon > 0\colon \exists n_0\colon \{x_{n_0}, x_{n_0+1}, \dots\} \subseteq B_\varepsilon(L).$$

A point L is a cluster point of a sequence $(x_i)_{i\in\mathbb{N}}$ iff

$$\forall\varepsilon > 0\colon \forall n_0\colon \exists n \geq n_0\colon d(x_n, L) < \varepsilon \qquad \text{or} \qquad \forall\varepsilon > 0\colon \forall n_0\colon \{x_{n_0}, x_{n_0+1}, \dots\} \cap B_\varepsilon(L) \neq \emptyset.$$

A sequence $(x_i)_{i\in\mathbb{N}}$ is *Cauchy* if $\qquad \forall\varepsilon > 0\colon \exists n_0\colon \forall n, m \geq n_0\colon d(x_n, x_m) < \varepsilon.$

A function f from (X, d) to another metric space (X', d') is *continuous* at $x \in X$ iff

$$\forall\varepsilon > 0\colon \exists\delta > 0\colon \forall y \in X\colon d(y, x) < \delta \Rightarrow d'(f(y), f(x)) < \varepsilon.$$

Equivalently, $\qquad f[B_\delta(x)] \subseteq B_\varepsilon(f(x)), \quad$ or $\quad B_\delta(x) \subseteq f^{-1}[B_\varepsilon(f(x))]$.

A function is *continuous* iff it continuous at every point x and *uniformly continuous* iff the δ can be chosen independently of x.

A set $U \subseteq X$ is *open* iff

$$\forall x \in U \colon \exists \varepsilon > 0 \colon \forall y \in X \colon d(y,x) < \varepsilon \Rightarrow y \in U \qquad \text{or} \qquad \forall x \in U \colon \exists \varepsilon > 0 \colon B_\varepsilon(x) \subseteq U.$$

Equivalently, U is open iff U is a union of open balls $U = \bigcup_x B_{\varepsilon_x}(x)$.

A point x is a point of closure of S iff $\quad \forall \varepsilon > 0 \colon B_\varepsilon(x) \cap S \neq \emptyset$.
The set of all points of closure of S is the *closure* of S, written $\bar{S}$. We say S is *closed* if $S = \bar{S}$ (it is always the case that $S \subseteq \bar{S}$). Note that the open ball $B_r(x)$ is always open and the closed ball $B'_r(x) = \{y : d(y,x) \leq r\}$ is always closed, however the closure of $B_r(x)$ may be smaller than $B'_r(x)$.

Define the distance between a point x and a set S by $\quad d(x,S) = \inf_{y \in S} d(x,y)$.
Then x is a point of closure of S iff $d(x,S) = 0$.

Lemma 1.

1. *X and $\emptyset$ are open.*
2. *Any finite intersection of open sets is open.*
3. *Any arbitrary union of open sets is open.*

Lemma 2.

1. *A set is closed iff its complement is open.*
2. *X and $\emptyset$ are closed. Any finite union of closed sets is closed. Any arbitrary intersection of closed sets is closed.*
3. *The closure of a set S is the smallest closed set containing S (= the intersection of all closed sets containing S).*

Lemma 3. *A function $f \colon X \to X'$ between two metric spaces is continuous iff $f^{-1}[U']$ is an open set in X for every open set $U' \subseteq X'$.*

Proof. If f is continuous and $x \in f^{-1}[U']$ then $f(x) \in U'$. But U' is open, so $B_\varepsilon(f(x)) \subseteq U'$ for some $\varepsilon > 0$. By the definition of continuity, $B_\delta(x) \subseteq f^{-1}[B_\varepsilon(x)] \subseteq f^{-1}[U']$ for some δ. Since this holds for all $x \in f^{-1}[U']$, $f^{-1}[U']$ is open.
Conversely, if $f^{-1}[U']$ is open for every open $U' \subseteq X'$, it must be open for $U' = B_\varepsilon(f(x))$. But $x \in f^{-1}[B_\varepsilon(f(x))]$, so $B_\delta(x) \subseteq f^{-1}[B_\varepsilon(f(x))]$ for some δ. Since this holds for any ε and any x, f is continuous. □

Subspaces and Products.

Given any metric space (X,d) and any subset $S \subseteq X$, we can view S as a metric space with (the restriction of) the same metric d. We call S a subspace of X.

Given two metric spaces (X_i, d_i), $i = 1,2$, one can construct many metrics on the Cartesian product $X_1 \times X_2$. For example, for $p \geq 1$ we can take $d_p((x_1,x_2),(y_1,y_2)) = (d(x_1,y_1)^p + d(x_2,y_2)^p)^{1/p}$, or $d_\infty((x_1,x_2),(y_1,y_2)) = \max\{d(x_1,y_1), d(x_2,y_2)\}$. Similarly for any finite number of factors. Starting with the usual metric on $\mathbb{R}$ this gives the various L^p metrics on $\mathbb{R}^n$.

Math 7351 2. Topological spaces Spring 2005

It turns out that many of the properties of metric spaces can be expressed entirely in terms of open sets (cf Lemmas 1–3 of the last section). This prompts the following definition.

A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X, called *open* sets, such that

T1. $\emptyset$ and X are open.

T2. Any finite intersection of open sets is open.

T3. An arbitrary union of open sets is open.

We *define* a set to be closed if its complement is open.
A *neighborhood* of a point x is a set S such that $x \in U \subseteq S$ for some open set U. We call S an *open neighborhood* if S is itself open and $x \in S$.

We can also define limits, continuity, closure etc. as follows:

A point L is a *limit* of $(x_i)_{i\in\mathbb{N}}$ iff	$\forall\,\text{nbhd}\,N$ of $x\colon \exists n_0\colon \{x_{n_0}, x_{n_0+1}, \dots\} \subseteq N$.
A point L is a *cluster point* of $(x_i)_{i\in\mathbb{N}}$ iff	$\forall\,\text{nbhd}\,N$ of $x\colon \forall n_0\colon \{x_{n_0}, x_{n_0+1}, \dots\} \cap N \neq \emptyset$.
$f\colon (X,\mathcal{T}) \to (X',\mathcal{T}')$ is *continuous* at x iff	$\forall\,\text{nbhd}\,N'$ of $f(x)\colon f^{-1}[N']$ is a nbhd of x.
$f\colon (X,\mathcal{T}) \to (X',\mathcal{T}')$ is *continuous* iff	$\forall\,\text{open}\,U'\colon f^{-1}[U']$ is open.
The *closure* $\bar{S}$ of S is	$\bar{S} = \bigcap_{F\supseteq S,\ F\text{ closed}} F$.
The *interior* $\mathring{S}$ of S is	$\mathring{S} = \bigcup_{U\subseteq S,\ U\text{ open}} U$.
The *boundary* ∂S of S is	$\partial S = \bar{S} \setminus \mathring{S}$.

Note: In general, the limit of a sequence may not be unique.

Exercises

1. Check these concepts agree with those defined for a metric space.
2. Check that a function is continuous iff it is continuous at x for all x.
3. Show that the composition of continuous functions is continuous (both 'at x' and generally).
4. Show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$ and $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$. Give an example when $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

Not all topologies can be derived from a metric. Also, different metrics can give rise to the same topology. We call two metrics d_1 and d_2 on a space X *equivalent* if they give rise to the same topology.

Lemma 1. *Two metrics d_1 and d_2 are equivalent iff for all x, any d_1-ball about x contains a d_2-ball about x and any d_2-ball about x contains a d_1-ball about x.*

Exercise: All L^p metrics on $\mathbb{R}^2$ are equivalent.

Although, many properties defined for metric spaces are topological, not all are. For example, uniform continuity and Cauchy sequences cannot be defined solely in terms of open sets. To see this, consider the two metrics $d_1(x,y) = |x-y|$ and $d_2(x,y) = |\frac{1}{x} - \frac{1}{y}|$ on $X = (0,\infty)$. These are equivalent, so give rise to the same topology, but $x_n = 1/n$ is Cauchy in d_1 but not in d_2, while $f(x) = x$ is uniformly continuous as a map $(X, d_1) \to (X, d_1)$, but not as a map $(X, d_1) \to (X, d_2)$ or $(X, d_2) \to (X, d_1)$.

The balls $B_r(x)$ play an important rôle in the study of metric spaces, so it is of interest to try and generalize the idea to arbitrary topological spaces.

Definition A collection $\mathcal{B}$ of subsets of X is a *base* for the topology $\mathcal{T}$, if every $B \in \mathcal{B}$ is open, and every open set is a union of elements of $\mathcal{B}$. Equivalently, if $x \in U$ and U is open then there is a $B \in \mathcal{B}$ with $x \in B \subseteq U$. A collection $\mathcal{B}_x$ is a *base of open neighborhoods of* x if every $B \in \mathcal{B}_x$ is an open neighborhood of x, and every open neighborhood of x contains an element of $\mathcal{B}_x$.

For a collection of sets $\mathcal{B}$ to form a base for some topology, it is enough that if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$. Similarly, if $\mathcal{B}_x$ is a base of open neighborhoods of x, then if $B_1, B_2 \in \mathcal{B}_x$, there is a $B_3 \in \mathcal{B}_x$ with $B_3 \subseteq B_1 \cap B_2$.

Clearly, the collection of all open balls (respectively all open balls about x) form a base (respectively base of open neighborhoods of x).

Definition If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on the same set X with $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say $\mathcal{T}_1$ is *weaker* or *coarser* than $\mathcal{T}_2$, or $\mathcal{T}_2$ is *stronger* or *finer* than $\mathcal{T}_1$.

The strongest possible topology on X is the *discrete* topology, where every subset of X is open. This topology is given by the discrete metric. The weakest possible topology on X is the *indiscrete* topology, where the only open sets are $\emptyset$ and X. This topology does not come from a metric if $|X| > 1$.

Since the intersection of any number of topologies is a topology (check this!), given any collection of sets $\mathcal{C}$, one can form the weakest topology in which all sets of $\mathcal{C}$ are open. One just intersects all the topologies that contain $\mathcal{C}$. More explicitly, the set of all finite intersections of sets from $\mathcal{C}$, $\mathcal{B} = \{\cap_{i=1}^n S_i : n \in \mathbb{N}, S_i \in \mathcal{C}\}$, is a base for a topology, and this topology is clearly the weakest topology containing $\mathcal{C}$.

If $f \colon X \to X'$ is an arbitrary function and $\mathcal{T}'$ is a topology on X', f induces a topology on X given by $f^*[\mathcal{T}'] = \{f^{-1}[U'] : U' \in \mathcal{T}'\}$. If X already has a topology $\mathcal{T}$, then the statement that f is continuous is equivalent to the statement that $f^*[\mathcal{T}']$ is weaker than $\mathcal{T}$. On the other hand, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X, then $\mathcal{T}_1$ is stronger than $\mathcal{T}_2$ iff the identity map $i \colon (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous.

Math 7351 3. New spaces from old Spring 2005

Homeomorphisms

Definition A *homeomorphism* is an invertible function between two topological spaces $f\colon (X,\mathcal{T}) \to (X',\mathcal{T}')$ such that both f and f^{-1} are continuous. Two spaces are *homeomorphic* if there is a homeomorphism between them.

Warning: If f is continuous and bijective, it does not follow that f^{-1} is continuous, e.g., $t \mapsto e^{it}$; $[0,2\pi) \to \{z \in \mathbb{C} : |z| = 1\}$.

Definition An *isometry* is an invertible function between two metric spaces $f\colon (X,d) \to (X',d')$ such that $d'(f(x),f(y)) = d(x,y)$. Two spaces are *isometric* if there is an isometry between them.

If two topological spaces are homeomorphic, then they can be considered to be the 'same' space with the points renamed. Similarly, two metric spaces are the 'same' if there is an isometry between them. Homeomorphisms preserve all 'topological' properties and isometries preserve all 'metric' properties.

Metric spaces also have the concept of *uniform homeomorphism*, where both f and f^{-1} are required to be *uniformly* continuous. Uniform homeomorphisms preserve some metric properties (e.g., Cauchy sequences, uniform continuity), but not others (e.g., boundedness of a metric). Two metrics d_1, d_2, on the same space are *uniformly equivalent* if the identity map $(X,d_1) \to (X,d_2)$ is a uniform homeomorphism.

Lemma 1. *Two metrics d_1 and d_2 are uniformly equivalent iff for all $\varepsilon > 0$ there is a $\delta > 0$ such that a d_1-ball of radius ε about any point x contains a d_2-ball of radius δ about x, and a d_2-ball of radius ε about any point x contains a d_1-ball of radius δ about x.*

Exercises

1. Show that the L^p metrics on $\mathbb{R}^n$ are all uniformly equivalent.
2. Show that any metric d is uniformly equivalent to the bounded metric $d'(x,y) = \min\{d(x,y),1\}$.
3. Show that $\mathbb{R}$ and $(0,1)$ are homeomorphic but not uniformly homeomorphic with the standard metric on both spaces.
4. Show that if $f\colon (X,d) \to (X',d')$ is uniformly continuous and (x_n) is a Cauchy sequence in X, then $(f(x_n))$ is a Cauchy sequence in X'.

Subspaces

Definition If $S \subseteq X$ we can define the *subspace* topology on S by declaring a subset of S open iff it is of the form $U \cap S$ where U is open in X.

Equivalently, it is the topology $i^*[\mathcal{T}]$ where $i\colon S \to X$ is the inclusion map, and is the weakest topology that makes i continuous. A set is closed in S iff it is of the form $F \cap S$ where F is closed in X.

Example $[0,1)$ is open in the subspace $[0,2)$ of $\mathbb{R}$ and closed in the subspace $[-1,1)$.

Exercise: Show that the topology given by the metric on a subspace of a metric space is the same as the subspace topology.

Product Spaces

Definition If $(X_\alpha, \mathcal{T}_\alpha)$, $\alpha \in I$ are topological spaces, we define the *product topology* on $X = \prod X_\alpha$ to be the weakest topology that makes all the projection maps $\pi_\alpha\colon X \to X_\alpha$ continuous. The collection $\{\prod U_\alpha : U_\alpha \in \mathcal{T}_\alpha, \text{ and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha\}$ forms a base for this topology.

For example, when there are only two factors $X_1 \times X_2$, the topology has a base consisting of *open rectangles* $U_1 \times U_2$, U_i open in X_i.

A sequence of points in a product space converges iff it converges in each coordinate separately. If all the X_α are equal, $X^I = \prod_{\alpha\in I} X$ can be thought of as the set of all functions $f\colon I \to X$. In this case, $f_n \to f$ in the product topology iff $f_n \to f$ pointwise, i.e., if $f_n(\alpha) \to f(\alpha)$ for all $\alpha \in I$.

Exercise: Show that for finite products of metric spaces, the 'L^p' product metrics all give rise to the product topology.

Lemma 2. *Given a countable collection of metric spaces (X_n, d_n) there is a metric on the product $X = \prod X_n$ which induces the product topology.*

Hint for proof: $d((x_n),(y_n)) = \sup_n\{\min\{d_n(x_n,y_n), 2^{-n}\}\}$ works. □

In general, the product topology on an uncountable product of metric spaces is not metrizable, i.e., it does not come from any metric.

Disjoint unions

Given any collection of disjoint topological spaces $(X_\alpha, \mathcal{T}_\alpha)$ we can form the disjoint union $\bigcup X_\alpha$ with the topology given by declaring a set U open iff each $U \cap X_\alpha$ is open in X_α. Equivalently, it is the strongest topology that makes all the inclusion maps $X_\alpha \to X$ continuous.

If each of the X_α are metric spaces with metrics d_α, one can define a metric on X which gives the disjoint union topology. For example, take $d(x,y) = 1$ if $x \in X_\alpha$, $y \in X_\beta$ and $\alpha \neq \beta$, and $d(x,y) = \min\{d_\alpha(x,y), 1\}$ if $x, y \in X_\alpha$.

Math 7351 4. Separation Spring 2005

Definition Two sets A and B are *separated* iff $A \cap \bar{B} = \bar{A} \cap B = \emptyset$.

Lemma 1. *The following all hold in any metric space*

Fréchet (T_1):	$\forall x, y,\ x \neq y \colon \exists$ *open* $U \colon x \in U,\ y \notin U$ ($\Leftrightarrow$ *points are closed).*
Hausdorff (T_2):	$\forall x, y,\ x \neq y \colon \exists$ *disjoint open* $U, V \colon x \in U,\ y \in V$.
Regular:	$\forall$ *closed* $A,\ y \notin A \colon \exists$ *disjoint open* $U, V \colon A \subseteq U,\ y \in V$.
Normal:	$\forall$ *disjoint closed* $A, B \colon \exists$ *disjoint open* $U, V \colon A \subseteq U,\ B \subseteq V$.
Completely Normal:	$\forall$ *separated* $A, B \colon \exists$ *disjoint open* $U, V \colon A \subseteq U,\ B \subseteq V$.
G_δ-space:	*Every closed set is countable intersection of open sets (a G_δ-set).*
1st Countable:	*There exists a countable base of neighborhoods about any point.*

Common notation: $T_3 = T_1$+Regular, $T_4 = T_1$+Normal, $T_5 = T_1$+Completely Normal.

Proof. (Sketch)
T_1: Take $U = B_\varepsilon(x)$ where $\varepsilon \leq d(x, y)$.
C.Normal: Set $U = \{x : d(x, A) < d(x, B)\}$ and $V = \{x : d(x, A) > d(x, B)\}$.
C.Normal$\Rightarrow$Normal$\Rightarrow$Regular$\Rightarrow T_2$: disjoint closed sets are separated and points are closed.
G_δ space: F closed $\Rightarrow F = \{x : d(x, F) = 0\} = \bigcap_{n=1}^\infty \{x : d(x, F) < 1/n\}$.
1st Countable: $\{B_{1/n}(x) : n = 1, 2, 3, \dots\}$ is a base at x. □

The following three results hold for any Normal space, and so in particular hold for metric spaces.

Theorem (Urysohn's Lemma) *If A and B are disjoint closed sets in a Normal space X then there exists a continuous function $f \colon X \to [0, 1]$ with $f = 0$ on A and $f = 1$ on B.*

[In a metric space we can set $f(x) = d(x, A)/(d(x, A) + d(x, B))$.]

Theorem (Tietze's Extension Theorem) *If A is a closed set in a Normal space X and $g \colon A \to \mathbb{R}$ is continuous, then there exists a continuous function $f \colon X \to \mathbb{R}$ with $f = g$ on A.*

Corollary *If $A_1, \dots, A_n$ are pairwise disjoint closed sets in a Normal space X, and $g_i \colon A_i \to \mathbb{R}$, $i = 1, \dots, n$, are continuous functions, then there exists a continuous function $f \colon X \to \mathbb{R}$ with $f = g_i$ on A_i.*

Definition A set S is *dense* if $\bar{S} = X$, or equivalently $S \cap U \neq \emptyset$ for all open $U \neq \emptyset$.

Definition An *open cover* is a collection of open sets whose union is X. It is finite, countable, etc., if the number of open sets in the cover is finite, countable, etc..

Lemma 2. *The following are equivalent for any metric space.*

Lindelöf:	*Every open cover has a countable subcover.*
Separable:	*There is a countable dense set of points.*
2nd Countable:	*There is a countable base for the topology.*

Proof. (Sketch)
Lind⇒Sep: $\{B_\varepsilon(x) : x \in X\}$ has a countable cover, $\{B_\varepsilon(x) : x \in S_\varepsilon\}$ say. Consider $\bigcup S_{1/n}$.
Sep⇒2nd C: S is a countable dense set $\Rightarrow \{B_{1/n}(x) : x \in S,\, n \in \mathbb{N}\}$ is a countable base.
2nd C⇒Lind: Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable base and $X = \bigcup U_\alpha$ an open cover. For each B_n pick (if possible) a U_{α_n} with $B_n \subseteq U_{\alpha_n}$. If $x \in X$, then $x \in U_\alpha$ for some α. But $\mathcal{B}$ is a base, so $x \subseteq B_n \subseteq U_\alpha$ for some n. Thus U_{α_n} exists and $x \in U_{\alpha_n}$. Hence the set of chosen U_{α_n} covers X. □

Examples

1. $\mathbb{R}^n$ is separable with any of the L^p metrics ($\mathbb{Q}^n$ is a countable dense subset).
2. $L^p(\mathbb{R})$ is separable if $p < \infty$ (approximate with 'rational' step functions).
3. $C([0,1])$ (continuous functions on $[0,1]$ with L^∞ norm) is separable (use uniform continuity, and approximate with piecewise linear functions with rational corners).
4. $L^\infty(\mathbb{R})$ and $C(\mathbb{R})$ are *not* separable (use 'Cantor Diagonal Argument': If $\{f_n\}$ is dense, construct bounded f which is far from f_n on $[n, n+1]$).

Lemma 3. *For metric spaces, these three equivalent conditions are preserved under taking (a) subspaces, (b) countable products, and (c) countable disjoint unions.*

Proof.
(a) Subspaces — use 2nd Countability: If $S \subseteq X$ and $\{B_i : i \in \mathbb{N}\}$ is a countable base for X, then $\{B_i \cap S : i \in \mathbb{N}\}$ is a countable base for S.
(b) Products — use Separability: Assume S_i is a countable dense subset of X_i, and fix $x_i \in S_i$. Then $S = \{(y_i) : y_i \in S_i$ and $y_i = x_i$ for all but finitely many $i\}$ is a countable dense subset of $X = \prod X_i$.
(c) Countable Unions — use Separability: Assume S_i is a countable dense subset of X_i, then $S = \bigcup S_i$ is a countable dense subset of $X = \bigcup X_i$. □

Metrizability

Question: When does a topology come from a metric?

This is in general a very difficult question, however the following result shows that a topology comes from a separable metric iff it is T_3 and 2nd Countable.

Theorem (Urysohn Metrization Theorem) *2nd Countable + T_3 ⇒ metrizable.*

[Urysohn actually proved '2nd Countable + T_4 ⇒ metrizable'. Later, Tychonoff noted that '2nd Countable + T_3 ⇒ T_4', so T_3 is enough.]

Math 7351 5. Completeness Spring 2005

Definition A metric space X is *complete* if every Cauchy sequence in X converges.

Examples $\mathbb{R}^n$, $L^p(S)$ $(1 \le p \le \infty)$, $C(S)$ with the L^∞-norm, $(S \subseteq \mathbb{R})$.

Lemma 1. *A closed subspace of a complete metric space is complete. If a subspace of a metric space is complete then it is closed.*

Hint: Any convergent sequence is Cauchy, and x is a limit of some sequence in S iff $x \in \bar{S}$.

Lemma 2. *A finite product of complete metric space is a complete metric space under any of the 'L^p' product metrics.*

Hint: A sequence is Cauchy/convergent iff it is Cauchy/convergent in each coordinate.

Theorem 1. *If (X, d) is a metric space, then there is a complete metric space $(\tilde{X}, \tilde{d})$ such that (X, d) is (isometric to) a dense subset of $(\tilde{X}, \tilde{d})$. Moreover, $(\tilde{X}, \tilde{d})$ is unique up to isometry, and satisfies the following universal property:*

> *If Y is a complete metric space and $f\colon X \to Y$ is uniformly continuous, then there exists a unique (uniformly) continuous map $\tilde{f}\colon \tilde{X} \to Y$ extending f to $\tilde{X}$.*

One construction is to take all Cauchy sequences in X and quotient out by the relation $(x_n) \sim (y_n)$ iff $\lim_{n\to\infty} d(x_n, y_n) = 0$. Then define $\tilde{d}(\overline{(x_n)}, \overline{(y_n)}) = \lim_{n\to\infty} d(x_n, y_n)$. The set X then corresponds to the subset of constant sequences.

Note that 'uniformly' is required in the above universal property. For example, $\tilde{f}$ does not exist if $X = \mathbb{Q}$, $Y = \tilde{X} = \mathbb{R}$ and $f(x) = 0$ for $x < \sqrt{2}$ and $f(x) = 1$ for $x > \sqrt{2}$.

Definition A metric space X is *totally bounded* if for any $\varepsilon > 0$ there is a finite set $\{x_1, \dots, x_n\}$ such that the $B_\varepsilon(x_i)$, $i = 1, \dots, n$, cover the whole of X.

Exercise: Show that any totally bounded space is bounded and separable. Give an example of a metric space that is bounded and separable but not totally bounded.

Theorem 2. *The following four conditions are equivalent for any metric space.*

Compact:	*Every open cover has a finite subcover.*
Sequentially compact:	*Every sequence has a convergent subsequence.*
Bolzano-Weierstrass property:	*Every sequence has a cluster point.*
The space is both Complete and Totally bounded.	

Proof. (Sketch)
Compact⇒T.B: $\{B_\varepsilon(x) : x \in X\}$ is an open cover, so has a finite subcover $\{B_\varepsilon(x_i) : i = 1, \dots, n\}$.
Compact⇒Complete: If (x_i) is a Cauchy sequence, $\forall \varepsilon > 0\colon \exists n_0\colon \forall n \ge n_0\colon d(x_n, x_{n_0}) < \varepsilon$.

Hence only finitely many points of the sequence lie in the open set $U_\varepsilon = \{x : d(x, x_{n_0}) > \varepsilon\}$. Thus no finite union of U_ε's can cover X. Hence the U_ε do not cover X. Pick $L \notin \bigcup_\varepsilon U_\varepsilon$. Then $\forall\varepsilon > 0\colon \exists n_0\colon \forall n \geq n_0\colon d(x_n, x_{n_0}) < \varepsilon$ and $d(L, x_{n_0}) \leq \varepsilon$, so $d(x_n, L) < 2\varepsilon$. Thus $x_n \to L$.

Complete+T.B.⇒Seq.C.: Take any sequence (x_i). By total boundedness, we can cover $X = \bigcup_{i=1}^{m_1} B_1(y_{1,i})$. At least one of these balls contains infinitely many terms $x_{n_{1,i}}$ of the sequence x_i. Now cover $X = \bigcup_{i=1}^{m_2} B_{1/2}(y_{2,i})$. At least one of these balls contains infinitely many terms $x_{n_{2,i}}$ of the sequence $x_{n_{1,i}}$. Repeat this process and consider the sequence $x_{n_{i,i}}$. Then for $j > i$, $x_{n_{j,j}} = x_{n_{i,k}}$ for some $k > i$ since the $x_{n_{j,*}}$ sequence is a subsequence of the $x_{n_{i,*}}$. Also $d(x_{n_{i,k}} - x_{n_{i,i}}) < 2/i$ since the $x_{n_{i,*}}$ sequence lies in the ball $B_{1/i}(y_{i,t})$ for some t. Thus $x_{n_{i,i}}$ is Cauchy, so converges by completeness.

Seq.C.⇒B-W: A limit of a subsequence is a cluster point of the sequence.

B-W⇒Compact: Suppose $\{U_\alpha\}$ is an open cover with no finite subcover. Pick any $x_1 \in X$ and choose U_{α_1} so that $B_{r_1}(x_1) \subseteq U_{\alpha_1}$ and r_1 is large, (say $> \frac{1}{2}\sup\{r : \exists\alpha\colon B_r(x_1) \subseteq U_\alpha\}$). Since there is no finite subcover, we can inductively choose $x_i \notin \bigcup_{j<i} U_{\alpha_j}$, r_i, and U_{α_i} similarly. The sequence x_i has a cluster point L. Since the U_α cover X, $L \in U_{\alpha_0}$ for some α_0. Since U_{α_0} is open, $B_{r_0}(L) \subseteq U_{\alpha_0}$ for some $r_0 > 0$. Pick an x_i with $d(x_i, L) < r_0/5$. But then $B_{4r_0/5}(x_i) \subseteq U_{\alpha_0}$, so $r_i > 2r_0/5$ by choice of U_{α_i}. But then all x_j, $j > i$, would be at least $r_0/5$ from L, and so L would not be a cluster point. □

Note that although the first three properties are topological, both completeness and total boundedness are both metric properties.

The following is an equivalent definition of compactness:

> If $\{F_\alpha\}$ is a collection of closed sets such that any finite intersection is non-empty, then the intersection of all the F_α is non-empty.

Lemma 3. *A closed subspace of a compact space is compact. If a subspace of a metric (or just T_2) space is compact then it is closed.*

Hints: 1st part: Add $X \setminus S$ to any open cover of S. 2nd part: If $x \in \bar{S} \setminus S$, consider the collection of open sets disjoint from some open neighborhood of x.

Theorem (Tychonoff) *A arbitrary product of compact topological spaces is compact.*

Lemma 4. *A continuous image of a compact space is compact.*

Proof. If $f\colon X \to Y$ and U_α cover Y, $f^{-1}[U_\alpha]$ cover X, take the U_α corresponding to a finite subcover of X. □

A subset of $\mathbb{R}$ is compact iff it is closed and bounded. Hence any continuous real valued function on a compact set is bounded and attains its bounds.

Lemma 5. *Any continuous map f from a compact metric space X to a metric space Y is uniformly continuous.*

Hint: Cover X with balls $B_{\delta_x}(x)$ where $f[B_{2\delta_x}(x)] \subseteq B_{\varepsilon/2}(f(x))$. Let $\delta = \min \delta_x$.

Math 7351 6. Baire Category Spring 2005

Theorem (Baire) *If $\{U_1, U_2, \dots\}$ is a countable collection of dense open subsets of a complete metric space X, then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.*

Proof. Fix any non-empty open U. We need to find an $L \in U \cap \bigcap_{n=1}^{\infty} U_n$. Pick a ball $B_{r_1}(x_1) \subseteq U$ (possible since U is a non-empty open set). Assume by induction that we have defined a ball $B_{r_n}(x_n)$. Since U_n is dense, there is a point $x_{n+1} \in U_n \cap B_{r_n}(x_n)$. Since $U_n \cap B_{r_n}(x_n)$ is open there exists $r_{n+1} > 0$ such that

$$\overline{B_{r_{n+1}}(x_{n+1})} \subseteq B_{2r_{n+1}}(x_{n+1}) \subseteq U_n \cap B_{r_n}(x_n).$$

W.l.o.g., $r_{n+1} < r_n/2$ so that $r_n \to 0$. Since $B_{r_{n+1}}(x_{n+1}) \subseteq B_{r_n}(x_n)$, $B_{r_m}(x_m) \subseteq B_{r_n}(x_n)$ for all $m > n$. Thus $d(x_m, x_n) < r_n$ for all $m > n$. Since $r_n \to 0$ as $n \to \infty$, (x_n) is a Cauchy sequence. Let $L = \lim x_n \in X$. Then since $x_m \in B_{r_{n+1}}(x_{n+1})$ for all $m > n+1$, $L \in \overline{B_{r_{n+1}}(x_{n+1})} \subseteq U_n$ for all n. Hence $L \in \bigcap_{n=1}^{\infty} U_n$. Also $L \in \overline{B_{r_2}(x_2)} \subseteq B_{r_1}(x_1) \subseteq U$. □

Definition A set S is *nowhere dense* if $\mathring{\bar{S}} = \emptyset$. Equivalently $\bar{S}$ contains no non-empty open set, or $X \setminus S$ contains an open dense set.

Definition A set S is *of 1st category* or *meager* if it is a countable union of nowhere dense sets. It is *of 2nd category* or *nonmeager* otherwise.

A subset of a nowhere dense (resp. 1st category) set is nowhere dense (resp. 1st category). Moreover, a countable union of 1st category sets is of 1st category.
A superset of a 2nd category set is of 2nd category.
One thinks of 1st category sets as being 'small' and 2nd category sets as being 'large'.

Corollary 1. *Any non-empty open subset of a complete metric space is of 2nd category.*

Proof. The complement of a countable union of nowhere dense sets contains a countable intersection of open dense sets. Therefore it is dense. Thus any non-empty open set meets the complement of any 1st category set, so cannot itself be of 1st category. □

Lemma 1. *If F is a closed set then $F \setminus \mathring{F}$ is nowhere dense. In particular a closed subset F of a complete metric space is of 1st category iff $\mathring{F} = \emptyset$.*

Proof. Since $F \setminus \mathring{F}$ is closed, it is enough to show it has no interior. Suppose $B_\varepsilon(x) \subseteq F \setminus \mathring{F}$ for some x and $\varepsilon > 0$. Now $x \notin \mathring{F}$, so $B_\varepsilon(x) \not\subseteq F$, contradicting the assumption that $B_\varepsilon(x) \subseteq F \setminus \mathring{F} \subseteq F$. For the second part, note if $\mathring{F} = \emptyset$ then $F = F \setminus \mathring{F}$, while if $\mathring{F} \neq \emptyset$ then $\mathring{F}$ is a non-empty open subset of X, thus $\mathring{F}$ (and hence F) is of 2nd Category. □

Corollary 2. *If $X \neq \emptyset$ is a complete space and F_n are closed subsets with $\bigcup_{n=1}^{\infty} F_n = X$, then at least one F_n has non-empty interior.*

Proof. If all $\mathring{F}_n = \emptyset$ then $X = \bigcup_{n=1}^{\infty} F_n$ is of 1st category. But X is open in X. □

Theorem (Uniform Boundedness Theorem) *Suppose $\mathcal{F}$ is a collection of continuous real-valued functions on a complete metric space X. Suppose that for all $x \in X$, $\sup_{f\in\mathcal{F}} |f(x)| < \infty$. Then there is a non-empty open set U and constant $M \in \mathbb{R}$ such that $\sup_{f\in\mathcal{F}} |f(x)| \le M$ for all $x \in U$.*

Proof. Let $F_n = \{x : \sup_{f\in\mathcal{F}} |f(x)| \le n\}$. Since each $f \in \mathcal{F}$ is continuous, $\{x : |f(x)| \le n\}$ is closed. Thus $F_n = \bigcap_{f\in\mathcal{F}}\{x : |f(x)| \le n\}$ is closed. Also $\bigcup_n F_n = X$. Thus there is an n with $\mathring{F}_n \ne \emptyset$. Let $M = n$ and $U = \mathring{F}_n$. □

Definition A topological space $(X, \mathcal{T})$ is *locally compact* if every $x \in X$ has a compact neighborhood, i.e., there exists a compact $K \subseteq X$ with $x \in \mathring{K}$ (note K need not be open).

Warning: There are several non-equivalent definitions in the literature. However, they are all equivalent for Hausdorff spaces.

A compact space is locally compact (take $K = X$), but some spaces are locally compact but not compact (e.g., $\mathbb{R}$). Amoung metric spaces, some complete spaces are not locally compact (e.g., $L^\infty(\mathbb{R})$) and some locally compact spaces are not complete (e.g., $(0, 1)$). Nevertheless, the Baire Category Theorem also holds for locally compact spaces.

Theorem (Locally compact BCT) *If $\{U_1, U_2, \dots\}$ is a countable collection of dense open subsets of a locally compact Hausdorff space X, then $\bigcap_{n=1}^\infty U_n$ is dense in X.*

Hint. First reduce to the case when $X = K$ is compact. Follow the original proof using arbitrary open V_n in place of $B_{r_n}(x_n)$. One needs the fact that if $W = V_n \cap U_n$ is open and $x \in W$ then there is an open V_{n+1} with $x \in V_{n+1}$, $\overline{V_{n+1}} \subseteq W$. This follows from the T_3 axiom (applied to x and $X \setminus W$) which holds in any compact Hausdorff space (prove this). Finally, use the finite intersection property of compact spaces to finish. □

Exercises

1. Show that $\mathbb{R}^n$ is not a countable union of $(n-1)$-dimensional hyperplanes.
2. Show that an infinite dimensional Banach space has uncountable dimension.
   [Hint: Show it is not a countable union of finite dimensional subspaces.]
3. Show that there is no continuous function that swaps the rationals and irrationals $f[\mathbb{Q}] \subseteq \mathbb{R} \setminus \mathbb{Q}$ and $f[\mathbb{R} \setminus \mathbb{Q}] \subseteq \mathbb{Q}$. [Hint: $f[\mathbb{R}]$ is countable and $\mathbb{R} = \bigcup_{z\in f[\mathbb{R}]} f^{-1}[z]$.]
4. Show that $\mathbb{Q}$ is not a G_δ-subset of $\mathbb{R}$.
5. Show that there is no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at all rational points and discontinuous at all irrational points.
6. Show that there is a subset of $[0, 1]$ which is of measure 1, but of 1st category.
7. Show there is a continuous function $\mathbb{R} \to \mathbb{R}$ that is not monotonic on any interval.
   [Hint: Show that $\{f \in C(\mathbb{R}) : f \text{ is monotonic on } [a, b]\}$ is nowhere dense in $C(\mathbb{R})$.]

Math 7351 7. Normed Spaces Spring 2005

Recall: A *normed space* is a real vector space X with a *norm* $\|.\|\colon X \to [0,\infty)$ with the following properties:

N1. If $\|v\| = 0$ then $v = 0$,

N2. $\|\lambda v\| = |\lambda|\|v\|$ where $\lambda \in \mathbb{R}$,

N3. $\|u + v\| \le \|u\| + \|v\|$.

If we drop condition N1 then we obtain a *pseudo-norm.*
A norm gives rise to a metric $d(x, y) = \|x - y\|$ and hence a topology on X.
Two norms, $\|.\|$ and $\|.\|'$, on X are (uniformly) equivalent if there exists constants K, K', such that for all v, $\|v\|' \le K\|v\|$ and $\|v\| \le K'\|v\|'$.
A *Banach Space* is a complete normed space.

Examples

1. $L^p(S)$, $S \subseteq \mathbb{R}$, are Banach spaces for all p, $1 \le p \le \infty$.
2. The finite l^p_n and infinite l^p sequence spaces are Banach spaces for all p, $1 \le p \le \infty$.
3. The completion of any normed space is a Banach space.
4. Any linear (vector) subspace of a normed space is a normed space. A linear subspace of a Banach space is Banach iff it is closed.
5. For $1 \le p < \infty$, the subspace $F = \{(x_n) : x_n = 0$ for all sufficiently large $n\}$ is a dense linear subspace of l^p.
6. Let $C^1([0, 1])$ be the space of continuously differentiable functions on $[0, 1]$ with norm $\|f\| = \sup_{x\in[0,1]}(|f(x)| + |f'(x)|)$. Then $C^1([0, 1])$ is a Banach space.

Lemma 1. *All norms on a finite dimensional space are equivalent. In particular, all finite dimensional normed spaces are complete.*

Proof. Since equivalence of norms is an equivalance relation, it is enough to show any $\|.\|$ is equivalent to $\|.\|_\infty$ on $X = \mathbb{R}^n$. Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{R}^n$ and $v = \sum \lambda_i e_i$. Then $\|v\| \le \sum_i |\lambda_i|\|e_i\| \le (\sum \|e_i\|) \max |\lambda_i| = K'\|v\|_\infty$ where $K' = \sum_{i=1}^n \|e_i\|$. Hence it is enough to show $\|v\|_\infty \le K\|v\|$. Assume otherwise and let v_n be a sequence with $\|v_n\|_\infty > n\|v_n\|$. By relacing v_n by $\|v_n\|_\infty^{-1} v_n$, we may assume $\|v_n\|_\infty = 1$. But the 'cube' $\{v : \|v\|_\infty = 1\}$ is compact, so there exists a $\|.\|_\infty$-cluster point v of the sequence v_n with $\|v\|_\infty = 1$. Now $v \ne 0$, so $\|v\| \ne 0$. Let $\varepsilon = \|v\|/2K'$ and pick v_n, n large, with $\|v_n - v\|_\infty < \varepsilon$. Then $\|v_n - v\| \le K'\|v_n - v\|_\infty < \|v\|/2$. Then $\|v_n\| > \|v\|/2$, which is a contradiction if n is large enough since $\|v_n\| < \|v_n\|_\infty/n = 1/n$.
Finally, all norms are complete since $\|.\|_\infty$ is, and completeness is preseved under uniform equivalence of metrics. □

Not all linear subspaces of a normed space are closed (e.g., example 5 above), however Lemma 1 implies that any *finite dimensional* linear subspace is closed.

Lemma 2. *If Y is a linear subspace of a normed space X then so is its closure $\bar{Y}$. Also, $\|v+Y\| = \inf_{y\in Y}\|v-y\|$ defines a pseudo-norm on the quotient space X/Y, which is a norm iff Y is closed.*

For $S \subseteq X$, let $\langle S\rangle$ denote the smallest linear subspace of X containing S, i.e., $\langle S\rangle = \{\sum_{i=1}^{n}\lambda_i v_i : n\in\mathbb{N},\, v_i\in S,\, \lambda_i\in\mathbb{R}\}$.

Linear operators

A map $T\colon X\to X'$ between two vector spaces is *linear* if $T(\lambda u+\mu v)=\lambda T(u)+\mu T(v)$ for all $u,v\in X$ and $\lambda,\mu\in\mathbb{R}$.

Definition If $(X,\|.\|)$ and $(X',\|.\|')$ are two (pseudo-)normed spaces and $T\colon X\to X'$ is linear, define $\|T\| = \inf\{M\ge 0 : \forall v\in X\colon \|T(v)\|'\le M\|v\|\} = \sup\{\|T(v)\|' : \|v\|\le 1\}$. We say that T is *bounded* if $\|T\|<\infty$.

Lemma 3 *If X and X' are normed spaces and $T\colon X\to X'$ is continuous at any point $v_0\in V$ then T is bounded. Conversely, if T is bounded then T is uniformly continuous.*

Proof. Pick $\varepsilon=1$, then $\exists\delta\colon \|v-v_0\|\le\delta \Rightarrow \|T(v)-T(v_0)\|'\le 1$. If $u\in V$, $u\ne 0$, let $v=v_0+(\delta/\|u\|)u$. Then $\|v-v_0\|=\delta$, so $1\ge\|T(v)-T(v_0)\|'=\|T(v-v_0)\|'=\|(\delta/\|u\|)T(u)\|'=(\delta/\|u\|)\|T(u)\|'$. Thus $\|T(u)\|'\le\delta^{-1}\|u\|$ and T is bounded.
If T is bounded and $\varepsilon>0$, let $\delta=\varepsilon/\|T\|$. Then if $\|u-v\|<\delta$, $\|T(u)-T(v)\|'=\|T(u-v)\|'\le\|T\|\|u-v\|<\varepsilon$. □

Definition An isomorphism between two normed spaces X and X' is a bounded linear map $T\colon X\to X'$ which has a bounded linear inverse $T^{-1}\colon X'\to X$. Equivalently, T is a linear homeomorphism.

Warning: It is not sufficient that T be a bijective bounded linear map.

Definition If X and Y are two normed spaces, then $\mathcal{B}(X,Y)$ is the set of all bounded linear maps from X to Y.

Lemma 4 *The set $\mathcal{B}(X,Y)$ is a vector space under addition $(T+T')(v)=T(v)+T'(v)$ and scalar multiplication $(\lambda T)(v)=\lambda T(v)$. With $\|T\|$ defined as above, it is also a normed space. If Y is a Banach space then so is $\mathcal{B}(X,Y)$.*

Proof. For the last part, if (T_n) is Cauchy, then so is $(T_n(v))$ for any $v\in X$. Define $T(v)=\lim T_n(v)$ and show $T\in\mathcal{B}(X,Y)$ and $\|T_n-T\|\to 0$. □

The space $X^*=\mathcal{B}(X,\mathbb{R})$ is called the *dual* space of X. Note that it is always a Banach space. The Riesz Representation Theorem of L^p-spaces gives an (isometric) isomorphism $L^p(S)^*\cong L^q(S)$, where $1\le p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Hence for $1<p<\infty$, $L^p(S)^{**}\cong L^p(S)$. However, it is not true in general that $X^{**}\cong X$, even for Banach spaces.

Math 7351 8. Hahn-Banach Theorem Spring 2005

Definition A convex functional is a map $p\colon X \to \mathbb{R}$ such that $p(\lambda x) = \lambda p(x)$ for $\lambda \geq 0$ and $p(x+y) \leq p(x) + p(y)$.

Theorem (Hahn-Banach) *Suppose X_0 is a linear subspace of X and p is a convex functional on X. If $T_0\colon X_0 \to \mathbb{R}$ is a linear map such that $T_0(x) \leq p(x)$ for all $x \in X_0$, then there exists an extension of T_0 to $T\colon X \to \mathbb{R}$ with $T(x) \leq p(x)$ for all $x \in X$.*

Proof. Let $\mathcal{X} = \{(T', X') : X'$ subspace of X, $X_0 \subseteq X'$, $T'|_{X_0} = T_0$, $\forall x \in X'\colon T'(x) \leq p(x)\}$. Partially order $\mathcal{X}$ by $(T', X') \leq (T'', X'')$ if $X' \subseteq X''$ and $T''|_{X'} = T'$. One can check the conditions of Zorn's lemma hold, so there is a maximal $(T_1, X_1) \in \mathcal{X}$. If $X_1 = X$ we are done. Otherwise choose $u \notin X_1$. Now

$$T_1(x) + T_1(x') = T_1(x + x') \leq p(x + x') \leq p(x + u) + p(x' - u)$$

So $T_1(x') - p(x' - u) \leq p(x + u) - T_1(x)$ and we can choose α so that

$$\sup_{x'}\{T_1(x') - p(x' - u)\} \leq \alpha \leq \inf_x\{p(x + u) - T_1(x)\}.$$

Define T_2 on $X_2 = \langle X_1, u\rangle$ by $T_2(x + \lambda u) = T_1(x) + \lambda\alpha$. Then for $\lambda \geq 0$, $T_2(x + \lambda u) \leq T_1(x) + \lambda(p(x/\lambda + u) - T_1(x/\lambda)) = p(x + \lambda u)$, $T_2(x - \lambda u) \leq T_1(x) - \lambda(T_1(x/\lambda) - p(x/\lambda - u)) = p(x - \lambda u)$. Thus $(T_2, X_2) \in \mathcal{X}$ and (T_1, X_1) was not maximal, a contradiction. □

Corollary *Suppose X_0 is a subspace of X and $T_0 \in X_0^*$. Then $\exists T \in X^*\colon \|T\|_{X^*} = \|T_0\|_{X_0^*}$.*

Proof. Take $p(x) = \|T_0\|\|x\|$. Use $-T(x) = T(-x)$ when bounding $|T(x)|$. □

Corollary *If $x \in X$, $x \neq 0$, then there is an $T \in X^*$ with $\|T\|_{X^*} = 1$ and $T(x) = \|x\|$.*

Proof. Define $T_0(\lambda x) = \lambda\|x\|$ on $X_0 = \langle x\rangle$. Extend to $T \in X^*$ with $\|T\| = \|T_0\| = 1$. □

Corollary *There is a natural isometric isomorphism between X and a subspace of X^{**}.*

Proof. Define for $x \in X$ a map $\mathrm{ev}_x\colon X^* \to \mathbb{R}$ by $\mathrm{ev}_x(T) = T(x)$. Clearly ev_x is a linear map $X^* \to \mathbb{R}$ and $|\mathrm{ev}_x(T)| = |T(x)| \leq \|x\|\|T\|$, so $\|\mathrm{ev}_x\| \leq \|x\|$ and $\mathrm{ev}_x \in X^{**}$. Define $T \in X^*$ so that $\|T\| = 1$ and $T(x) = \|x\|$. Then $|\mathrm{ev}_x(T)| = \|x\| = \|x\|\|T\|$, so $\|\mathrm{ev}_x\| = \|x\|$. The map $x \mapsto \mathrm{ev}_x$ is linear, so is an isometry between X and a subspace of X^{**}. □

Note: One can define the completion of a normed space as the closure of the image of X in X^{**}.

Definition X is called *reflexive* if the image of X is the whole of X^{**}.

Example If $X = L^1(\mathbb{R})$ then X^{**} is strictly larger than the image of X: By the Riesz Representation Theorem we know $X^* \cong L^\infty(\mathbb{R})$, where $g \in L^\infty$ corresponds to the functional $f \mapsto \int gf$. Consider $T_0\colon C(\mathbb{R}) \to \mathbb{R}$ given by $T_0(g) = g(0)$. Then $\|T_0\| = 1$. Extend T_0 to $T\colon L^\infty(\mathbb{R}) \to \mathbb{R}$. Then $T \in X^{**}$, but is not of the form ev_f for any $f \in L^1(\mathbb{R})$, since $\mathrm{ev}_f(g) = \int gf$ does not agree with T_0 on $C(\mathbb{R})$ for any f.

Math 7351 9. Closed Graph Theorem Spring 2005

Theorem (Open Mapping Theorem) *If X and Y are Banach spaces and $T \in \mathcal{B}(X,Y)$ is surjective, then T is an* open mapping, *i.e., $T[U]$ is open in Y for every open $U \subseteq X$.*

Proof. Since $X = \bigcup_n B_n(0)$, $Y = T[X] = \bigcup_n T[B_n(0)]$. Thus, by the Baire Category Theorem, some $T[B_n(0)]$ is not nowhere dense. Pick $B_\varepsilon(y) \subseteq \overline{T[B_n(0)]}$. Then $\exists z \in T[B_n(0)]$: $z \in B_{\varepsilon/2}(y)$, so $B_{\varepsilon/2}(0) \subseteq \overline{T[B_n(0)] - z} \subseteq \overline{T[B_{2n}(0)]}$. Fix $y_0 \in B_{\varepsilon/2}(0) \subseteq Y$ and choose $x_0 \in B_{2n}(0) \subseteq X$ so that $\|y_0 - T(x_0)\| < \varepsilon/4$. Write $y_1 = 2(y_0 - T(x_0)) \in B_{\varepsilon/2}(0)$ and repeat, inductively defining $y_{n+1} = 2(y_n - T(x_n)) \in B_{\varepsilon/2}(0) \subseteq Y$ and $x_n \in B_{2n}(0) \subseteq X$. Now $\|y_0 - T(\sum_{i=0}^{n-1} 2^{-i}x_i)\| = \|y_n/2^n\| < \varepsilon/2^n$. But $\sum_{i=0}^{n-1} 2^{-i}x_i$ is Cauchy, so $x = \sum_{i=0}^{\infty} 2^{-i}x_i \in B_{4n}(0)$ exists. Since T is bounded, $\|T(x) - \lim T(\sum_{i=0}^{n-1} 2^{-i}x_i)\| \to 0$, so $y_0 = T(x) \in T[B_{4n}(0)]$. Since this holds for any y_0, $B_{\varepsilon/2}(0) \subseteq T[B_{4n}(0)]$. Thus by linearity, if $x \in U$ then $T[U]$ contains a neighborhood of $T[x]$. □

Theorem (Inverse Mapping Theorem) *If X and Y are Banach spaces and $T \colon X \to Y$ is a bijective bounded linear map, then T^{-1} is also bounded (so T is an isomorphism).*

Proof. If T is bijective and open, then T^{-1} is continuous. □

Corollary *If X is complete with respect to $\|.\|$ and $\|.\|'$ and $\|.\| \leq K\|.\|'$, then $\|.\|$ and $\|.\|'$ are equivalent.*

Proof. The identity map $(X, \|.\|') \to (X, \|.\|)$ is bijective and bounded. □

Theorem (Closed Graph Theorem) *Suppose X and Y are Banach spaces, $T \colon X \to Y$ is linear, and for any sequence $x_n \in X$ with x_n and $T(x_n)$ both convergent, $T(\lim x_n) = \lim T(x_n)$. Then T is bounded.*

[The condition on T states that $\{(x, T(x)) : x \in X\}$ is a closed subset of $X \times Y$.]

Proof. Define a new norm on X by $|||x||| = \|x\| + \|T(x)\|'$. If (x_n) is Cauchy in $|||.|||$ then both (x_n) and $T(x_n)$ are Cauchy in X and Y, so $x = \lim x_n$ and $y = \lim T(x_n)$ exists. By hypothesis, $y = T(x)$. Thus $|||x_n - x||| = \|x_n - x\| + \|T(x_n) - T(x)\|' \to 0$. Hence $(X, |||.|||)$ is complete. But $\|.\| \leq 1|||.|||$, so $\|T(x)\|' \leq |||x||| \leq K\|x\|$ and T is bounded. □

Theorem (Uniform Boundedness Theorem) *If X is a Banach space and Y is a normed space, and $\mathcal{F} \subseteq \mathcal{B}(X,Y)$ is a collection of bouned linear maps such that for all $x \in X$, $\sup_{T\in\mathcal{F}} \|T(x)\| < \infty$. Then $\sup_{T\in\mathcal{F}} \|T\| < \infty$.*

Proof. Applying the Uniform Boundedness Theorem of section 6 to $\{\|T(.)\| : T \in \mathcal{F}\}$ there is $M \in \mathbb{R}$ and open $U \neq \emptyset$ in X such that $\|T(x)\| \leq M$ for all $T \in \mathcal{F}$ and $x \in U$. Pick $B_\varepsilon(x_0) \subseteq U$. Then for $\|x\| \leq 1$, $\|T(x)\| = \|\frac{2}{\varepsilon}T(x_0 + \frac{\varepsilon}{2}x) - \frac{2}{\varepsilon}T(x_0))\| \leq 4M/\varepsilon$. Thus $\|T\| \leq 4M/\varepsilon$ for all $T \in \mathcal{F}$. □

Math 7351 10. Hilbert spaces Spring 2005

Definition An *inner product* on a real or complex vector space X is a map $(\,,\,)\colon X\times X\to\mathbb{R}$ or $X\times X\to\mathbb{C}$ such that:

I1. $(\lambda x_1+\mu x_2, y)=\lambda(x_1,y)+\mu(x_2,y)$,

I2. $(x,y)=\overline{(y,x)}$,

I3. $(x,x)\geq 0$ and $(x,x)=0$ iff $x=0$.

If $(\,,\,)$ is an inner product on V, define $\|x\|=\sqrt{(x,x)}$.

Lemma 1.

1. *$\|.\|$ is a norm on V,*
2. *$|(x,y)|\leq\|x\|\|y\|$ (Cauchy-Schwarz)*
3. *$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$ (Parallelogram law)*
4. *$4(x,y)=\sum_{\zeta\in\{\pm1,\pm i\}}\zeta\|x+\zeta y\|^2$ (Polarization identity)*
5. *$2(x,y)=\|x+y\|^2-\|x\|^2-\|y\|^2$ for real spaces (Polarization identity)*

Definition A *Euclidean space* is a normed space with norm given by an inner product. A *Hilbert space* is a complete Euclidean space (i.e., Banach space given by inner product).

Exercise: The spaces $L^2(S)$ and l^2 are Hilbert spaces with $(f,g)=\int f\bar{g}$ and $((x_n),(y_n))=\sum x_n\bar{y}_n$ respectively.

Definition Two sets S_1 and S_2 are *orthogonal* $S_1\perp S_2$, if $(x,y)=0$ for all $x\in S_1$, $y\in S_2$. The *orthogonal complement* of S is $S^\perp=\{x:(x,y)=0 \text{ for all } y\in S\}$.

Lemma 2. *$S^\perp$ is a closed linear subspace of X.*

Theorem (Projection Theorem) *If F is a complete subspace of a Euclidean space X, then every element $x\in X$ has a unique representation as $x=x_\parallel+x_\perp$, $x_\parallel\in F$, $x_\perp\in F^\perp$.*

Hint. Let $d=d(x,F)$ and choose $x_n\in F$ with $\|x_n-x\|^2\leq d^2+\frac{1}{n}$. Use Parallelogram law to show (x_n) Cauchy and set $x_\parallel=\lim x_n$, $x_\perp=x-x_\parallel$. □

Corollary *If F is a closed subspace of a Hilbert space then $(F^\perp)^\perp=F$.*

Theorem (Riesz Representation Theorem) *If X is a Hilbert space, then every $T\in H^*$ can be given by $T(x)=(x,y)$ for some unique $y\in H$. Moreover $\|T\|=\|y\|$.*

Hint. Let $F=T^{-1}[\{0\}]$ and choose $y\in F^\perp$ with $T(y)=\|y\|^2$. □

Note: This shows that there is a natural anti-isomorphism between H and H^* (an anti-isomorphism since λy corresponds to $\bar{\lambda}T$). Also, every Hilbert space is reflexive.

Definition An *orthonormal system* is a set $S \subseteq X$ such that $(e, e') = 0$ for all $e, e' \in S$, $e \neq e'$, and $(e, e) = 1$ for all $e \in S$. An orthonormal system is *complete* if it is maximal, i.e., there is no orthonormal system containing S as a proper subset.

Note: By Zorn's lemma, every Euclidean space has a complete orthonormal system.
An orthonormal system is complete iff $\langle S \rangle^\perp = \{0\}$.

Theorem (Gram-Schmidt orthogonalization) *If $\{x_1, x_2, \dots\}$ is a linearly independent sequence in a Euclidean space, then there exists an orthonormal system $\{e_1, e_2, \dots\}$ with $\langle e_1, \dots, e_n \rangle = \langle x_1, \dots, x_n \rangle$ for every n.*

Proof. Let $z_n = x_n - \sum_{i=1}^{n-1} (x_n, e_i) e_i$ and set $e_n = z_n / \|z_n\|$. ◻

Corollary *Every separable Euclidean space has a countable complete orthonormal system.*

Proof. Take a countable dense set $\{x_1, x_2, \dots\}$. Form an independent subsequence (x_{n_k}) by choosing n_k minimal so that $x_{n_k} \notin \langle x_{n_1}, \dots, x_{n_{k-1}} \rangle$. Then $\{x_{n_1}, \dots\}$ is an independent set and $\langle x_{n_1}, \dots \rangle$ is dense. Apply Gram-Schmidt to $\{x_{n_1}, x_{n_2}, \dots\}$ to get orthonormal $\{e_1, e_2, \dots\}$. Assume $\{y\} \cup \{e_1, \dots\}$ is an orthonormal system. There is an n such that $x_n \in B_1(y)$, so $B_1(y) \cap \langle x_{n_1}, \dots, x_{n_k} \rangle \neq \emptyset$ when $n_{k+1} > n$. Hence $B_1(y) \cap \langle e_1, \dots, e_k \rangle \neq \emptyset$. But if $y' \in \langle e_1, \dots, e_k \rangle$ then $y' \perp y$, so $\|y' - y\|^2 = \|y\|^2 + \|y'\|^2 \geq 1$, so $y' \notin B_1(y)$, a contradiction. ◻

Lemma 3. *Let $S = \{e_1, \dots\}$ be an orthonormal system in a Hilbert space X, and let $F = \overline{\langle S \rangle}$. Write $x \in X$ as $x = x_\parallel + x_\perp$ with $x_\parallel \in F$, $x_\perp \in F^\perp$. Then $\sum_n (x, e_n) e_n$ converges to $x_\parallel$ and $\sum_n |(x, e_n)|^2 = \|x_\parallel\|^2 \leq \|x\|^2$.*

Proof. Write $c_i = (x, e_i)$ and $x_n = \sum_{i=1}^n c_i e_i$. Then $(x - x_n) \perp e_i$ for $i \leq n$, so $(x - x_n) \perp x_n$. Thus $\|x\|^2 = \|x_n\|^2 + \|x - x_n\|^2 \geq \sum_{i=1}^n |c_i|^2$. Thus $\sum_{i=1}^\infty |c_i|^2 < \infty$. Now $\|x_n - x_m\|^2 = \sum_{i=n+1}^m |c_i|^2 \leq \sum_{i=n+1}^\infty |c_i|^2 \to 0$ as $n \to \infty$. Thus x_n is Cauchy in the complete space F. Let $x' = \lim x_n \in F$. Then $(x - x', e_i) = \lim_n (x - x_n, e_i) = 0$ for all i, so $(x - x')^\perp \supseteq S$. But $(x - x')^\perp$ is a closed linear subspace, so $(x - x')^\perp \supseteq F$. Thus $x - x' \in F^\perp$ and $x' = x_\parallel$. Finally, $\sum_{i=1}^\infty |c_i|^2 = \lim \sum_{i=1}^n |c_i|^2 = \lim \|x_n\|^2 = \|x_\parallel\|^2$. ◻

Definition We call $c_i = (x, e_i)$ the *Fourier coefficients of x with respect to the orthonormal system* $\{e_1, \dots\}$.

Theorem *Any separable Hilbert space is isometric to either the n-dimenensional Euclidean space l_n^2 or the l^2 sequence space.*

Proof. Let $\{e_1, e_2, \dots\}$ be a complete orthonormal system. Map $x \in X$ to $(c_i) \in l^2$ where $c_i = (x, e_i)$ are the Fourier coefficients of x. ◻

Example The orthonormal system $\{\sqrt{2} \sin 2n\pi x : n \geq 1\} \cup \{\sqrt{2} \cos 2n\pi x : n \geq 1\} \cup \{1\}$ gives an isometry between $L^2([0, 1])$ and l^2.