7281 Homework Assignment 3 Fall 2007

Due Tuesday, November 6th.

1. Let π be a permutation of the numbers $1, 2, \ldots, n$. Show that

$$\operatorname{sgn}(\pi) = \prod_{1 \le i < j \le n} \frac{\pi(j) - \pi(i)}{j - i}$$

   [Hint: it is enough to show that this expression changes sign when we swap the values of $\pi(k)$ and $\pi(k+1)$ for any k.]

2. Matrices A and B are *similar* if there is an invertible matrix P such that $B = PAP^{-1}$. Show that if A and B are similar then they have the same determinant.

3. A matrix A is *nilpotent* if $A^n = 0$ for some positive integer n.

   (a) Give an example of a non-zero nilpotent matrix A.

   (b) If A is nilpotent, what is $\det(A)$?

4. Show that

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{bmatrix} = \prod_{1 \le i < j \le n} (a_j - a_i)$$

   [Hint: try the $n = 3$ and $n = 4$ cases first.]

5. The points $(0, 0)$, $(1, 3)$, $(5, 2)$, and $(7, 8)$ form a quadrilateral Q. Find the area of Q by first applying the transformation $\begin{bmatrix} 2 & -5 \\ -3 & 1 \end{bmatrix}$ to Q.