Let $A$ be an $n \times n$ matrix. Define a minimal polynomial of $A$ to be a non-zero polynomial $m(x)$ of smallest degree such that $m(A) = 0$. (Since multiplying $m(x)$ by a non-zero constant does not affect the definition, we may assume the leading term of $m(x)$ has coefficient 1.)

1. (a) Show that if $h(x)$ is a polynomial with $h(A) = 0$ then $h(x) = m(x)q(x)$ for some polynomial $q(x)$.
   (b) Deduce that $m(x)$ is unique (up to multiplication by a constant).
   (c) If $A^2 = A$, what are the possibilities for $m(x)$?

2. (a) Show that if $m(\lambda) = 0$ then $\lambda$ is an eigenvalue of $A$.
   [Hint: Use Cayley-Hamilton and Question 1.]
   (b) Show that if $\lambda$ is an eigenvalue of $A$ then $m(\lambda) = 0$.
   [Hint: apply $m(A)$ to an eigenvector of $A$.]

3. Recall that a matrix is nilpotent if $A^N = 0$ for some $N$. What is the characteristic polynomial of a nilpotent matrix?

4. (a) Show that the minimal polynomial of $A$ and $P^{-1}AP$ are the same.
   (b) Show that if $A$ is diagonalizable then the minimal polynomial has distinct roots.
   (c) Show that if $A$ has distinct roots then $A$ is diagonalizable. [Hint: it is enough to show that every vector is a linear combination of eigenvectors.]

5. (a) Find the eigenvalues of the matrix
   \[
   \begin{bmatrix}
   0 & 1 & 0 \\
   -4 & 4 & 0 \\
   -3 & 1 & 3 \\
   \end{bmatrix}
   \]
   (b) For each of the eigenvalues in (a), find a basis for the corresponding eigenspace, together with the geometric and algebraic multiplicities of the eigenvalue.