Math 7350 Homework Assignment 1 Fall 2004

Due September 16.

1. Let $A_{i,j}$ be subsets of a set X for $i, j \in \mathbb{N}$. Show that
$$\bigcap_{i=0}^{\infty} \bigcup_{j=0}^{\infty} A_{i,j} = \bigcup_{(a_i)} \bigcap_{i=0}^{\infty} A_{i,a_i}$$
where the second union is over all sequences $(a_i)_{i=0}^{\infty}$ of natural numbers.

2. Let $\mathcal{C}$ be a collection of subsets of a set X, and let $\mathcal{A}$ be the algebra generated by $\mathcal{C}$. Show that the σ-algebra generated by $\mathcal{C}$ is equal to the σ-algebra generated by $\mathcal{A}$.

3. Give an example of a partial order with a unique minimal element, but no smallest element.

4. Assume A and B are well ordered sets, with well orderings $\leq$ and $\leq'$ respectively. If A is order isomorphic to an initial segment of B and B is order isomorphic to an initial segment of A, show that A and B are order isomorphic. [Hint: show that the composition of the two order isomorphisms is the identity.]

5. We call a set X *Dedekind finite* if every injection $f\colon X \to X$ is also surjective.
   (a) Show that if X is not Dedekind finite, then there is an injection $g\colon \mathbb{N} \to X$. [Hint: If $x_0 \notin f[X]$, $x_{n+1} = f(x_n)$, and f is injective, then the x_n are distinct.]
   (b) Show that if there exists an injection $g\colon \mathbb{N} \to X$ then X is not Dedekind finite.

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1. Let $A_{i,j}$ be subsets of a set X for $i, j \in \mathbb{N}$. Show that
$$\bigcap_{i=0}^{\infty} \bigcup_{j=0}^{\infty} A_{i,j} = \bigcup_{(a_i)} \bigcap_{i=0}^{\infty} A_{i,a_i}$$
where the second union is over all sequences $(a_i)_{i=0}^{\infty}$ of natural numbers.

   Assume $x \in \bigcap_{i=0}^{\infty} \bigcup_{j=0}^{\infty} A_{i,j}$. Then, for all i, $x \in \bigcup_{j=0}^{\infty} A_{i,j}$. Hence, if we fix i, there is a j such that $x \in A_{i,j}$. Set a_i to be the smallest such j. Now $x \in A_{i,a_i}$ for all i, so $x \in \bigcap_{i=0}^{\infty} A_{i,a_i}$. In particular, $x \in \bigcup_{(a_i)} \bigcap_{i=0}^{\infty} A_{i,a_i}$.

   Now assume $x \in \bigcup_{(a_i)} \bigcap_{i=0}^{\infty} A_{i,a_i}$. Then there exists a sequence $(a_i)_{i=0}^{\infty}$ such that $x \in \bigcap_{i=0}^{\infty} A_{i,a_i}$, so $x \in A_{i,a_i}$ for all i. But then, for all i, $x \in \bigcup_j A_{i,j}$. Hence $x \in \bigcap_{i=0}^{\infty} \bigcup_{j=0}^{\infty} A_{i,j}$.

   By Extensionality, the two sets are equal.

1. Let $\mathcal{C}$ be a collection of subsets of a set X, and let $\mathcal{A}$ be the algebra generated by $\mathcal{C}$. Show that the σ-algebra generated by $\mathcal{C}$ is equal to the σ-algebra generated by $\mathcal{A}$.

   Let $\sigma(\mathcal{C})$ be the σ-algebra generated by $\mathcal{C}$ and let $\sigma(\mathcal{A})$ be the σ-algebra generated by $\mathcal{A}$. Since any σ-algebra is also an algebra, $\sigma(\mathcal{C})$ is an algebra containing $\mathcal{C}$, and hence contains $\mathcal{A}$, the smallest algebra containing $\mathcal{C}$. Now $\sigma(\mathcal{C})$ is a σ-algebra containing $\mathcal{A}$, so contains $\sigma(\mathcal{A})$, the smallest such σ-algebra. Similarly $\sigma(\mathcal{A})$ is a σ-algebra that contains $\mathcal{A}$, and hence $\mathcal{C}$, and so contains $\sigma(\mathcal{C})$. Thus $\sigma(\mathcal{C}) = \sigma(\mathcal{A})$.

2. Give an example of a partial order with a unique minimal element, but no smallest element.

   Let $X = \mathbb{Z} \cup \{\star\}$ with the ordering given by the usual ordering on $\mathbb{Z}$, $\star \le \star$, but $\star$ unrelated to any element of $\mathbb{Z}$. Since the usual ordering on $\mathbb{Z}$ is a partial ordering, we only need to check the axioms involving $\star$, which are trivial. Now $\star$ is clearly minimal since $x \le \star$ implies $x = \star$. On the other hand, no $x \in \mathbb{Z}$ is minimal since $x - 1 < x$. Finally, X has no smallest element, since the minimal element $\star$ is not $\le$ every other element (or indeed any other element).

3. Assume A and B are well ordered sets, with well orderings $\le$ and $\le'$ respectively. If A is order isomorphic to an initial segment of B and B is order isomorphic to an initial segment of A, show that A and B are order isomorphic. [Hint: show that the composition of the two order isomorphisms is the identity.]

Suppose $f\colon A \to B'$ and $g\colon B \to A'$ are order isomorphisms where A' and B' are initial segments of A and B respectively. Consider the function $h = g \circ f\colon A \to A$. Assume h is not the identity on A and let $x = \min\{y \in A : h(y) \neq y\}$. There are two possible cases:

Case 1: $h(x) < x$.
In this case $h(h(x)) = h(x)$ (by minimality of x). But then $g(f(h(x))) = g(f(x))$, so $f(h(x)) = f(x)$ (injectivity of g) and $h(x) = x$ (injectivity of f). This contradicts $h(x) < x$.

Case 2: $h(x) > x$.
Now if $y \geq x$ then $f(y) \geq f(x)$ and $g(f(y)) \geq g(f(x))$ by the order preserving property of f and g. Hence $h(y) \geq h(x) > x$ for all $y \geq x$ and $h(y) = y < x$ for all $y < x$. Hence $x \notin h[A]$. But $g[B]$ is an initial segment of A and contains $h(x) = g(f(x)) > x$, so $x = g(x')$ for some $x' \in B$. But now $f(x) > x'$ by the order preserving property of g. Now since $f[A]$ is an initial segment, $x' = f(x'')$ for some $x'' \in A$. But then $h(x'') = x$, a contradiction.

Now $h(x) = x$ for all x, so $A' = g[B] \supseteq g[f[A]] = A$. Hence $A' = A$ and g is an order isomorphism from B to A.

4. We call a set X *Dedekind finite* if every injection $f\colon X \to X$ is also surjective.

   (a) Show that if X is not Dedekind finite, then there is an injection $g\colon \mathbb{N} \to X$. [Hint: If $x_0 \notin f[X]$, $x_{n+1} = f(x_n)$, and f is injective, then the x_n are distinct.]

   Since X is not Dedekind finite, there exists an injective function $f\colon X \to X$ such that $f[X] \neq X$. Pick $x_0 \in X \setminus f[X]$ and define recursively $x_{n+1} = f(x_n)$ for $n \geq 0$. We shall show by induction on n that $x_n \neq x_m$ for all $m < n$. Clearly this holds for $n = 0$ since no such m exists. Assume true for n and consider $x_{n+1} = f(x_n)$. Clearly $x_{n+1} \neq x_0$ since $x_0 \notin f[X]$. However, if $m > 0$ then by induction $x_n \neq x_{m-1}$. But f is injective, so $x_{n+1} = f(x_n) \neq f(x_{m-1}) = x_m$ as required.

   Now define $g(n) = x_n$, so that $g\colon \mathbb{N} \to X$ is an injective function.

   (b) Show that if there exists an injection $g\colon \mathbb{N} \to X$ then X is not Dedekind finite.

   Define $f\colon X \to X$ by

$$f(x) = \begin{cases} x & \text{if } x \notin g[\mathbb{N}]; \\ g(n+1) & \text{if } x = g(n) \in g[\mathbb{N}]. \end{cases}$$

   Clearly f is injective, but $g(0) \notin f[X]$.