Math 7350 Homework Assignment 2 Fall 2004

Due September 30.

1. Let $\mathcal{C}$ be a collection of subsets of a set X, and let $\mathcal{A}$ be the σ-algebra generated by $\mathcal{C}$. Show that if $A \in \mathcal{A}$, then there exists a countable subset $\mathcal{C}_0 \subseteq \mathcal{C}$ such that A is in the σ-algebra generated by $\mathcal{C}_0$. [Hint: show that the union of all the σ-algebras generated by the countable subsets of $\mathcal{C}$ is a σ-algebra.]

2. Show that there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$, but $f(x)$ is *not* of the form $f(x) = \lambda x$. [Hint: Consider $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$ and define a suitable f using a basis for this vector space.]

3. (a) If (x_n) and (y_n) are two real sequences, show that
$$\overline{\lim}\, x_n + \underline{\lim}\, y_n \le \overline{\lim}(x_n + y_n) \le \overline{\lim}\, x_n + \overline{\lim}\, y_n$$
provided both sides are not of the form $\infty - \infty$.

   (b) Give examples to show that both inequalities may be strict.

4. Let S be a set of positive real numbers. Define
$$\textstyle\sum_{x\in S} x = \sup \left\{ \sum_{x \in S_0} x : S_0 \text{ is a finite subset of } S \right\}.$$
Show that if S is uncountable then $\sum_{x\in S} x = +\infty$.

5. Let $(x_n)_{n=0}^{\infty}$ be a real sequence, and for all i, let y_i be a cluster point of $(x_n)_{n=0}^{\infty}$. Show that any cluster point of $(y_i)_{i=0}^{\infty}$ is also a cluster point of $(x_n)_{n=0}^{\infty}$.

Math 7350 Homework 2 Solutions Fall 2004

1. Let $\mathcal{C}$ be a collection of subsets of a set X, and let $\mathcal{A}$ be the σ-algebra generated by $\mathcal{C}$. Show that if $A \in \mathcal{A}$, then there exists a countable subset $\mathcal{C}_0 \subseteq \mathcal{C}$ such that A is in the σ-algebra generated by $\mathcal{C}_0$. [Hint: show that the union of all the σ-algebras generated by the countable subsets of $\mathcal{C}$ is a σ-algebra.]

   If $\mathcal{C}_0$ is a countable subset of $\mathcal{C}$, then $\mathcal{A} = \sigma(\mathcal{C})$ contains $\mathcal{C}_0$, is a σ-algebra, and so contains $\sigma(\mathcal{C}_0)$, the σ-algebra generated by $\mathcal{C}_0$. Let $U = \bigcup_{\mathcal{C}_0} \sigma(\mathcal{C}_0)$. Then $U \subseteq \mathcal{A}$. If $x \in \mathcal{C}$ then $x \in \sigma(\{x\}) \subseteq U$, so $\mathcal{C} \subseteq U$. To prove $U = \mathcal{A}$, it is therefore enough to show that U is a σ-algebra.

   (1) $\emptyset \in \sigma(\emptyset) \subseteq U$.
   (2) If $X_1, X_2, \dots \in U$, then there are countable sets $\mathcal{C}_i \subseteq \mathcal{C}$ with $X_i \in \sigma(\mathcal{C}_i)$. Hence $X_i \in \sigma(\bigcup \mathcal{C}_i)$, and since $\sigma(\bigcup \mathcal{C}_i)$ is a σ-algebra, $\bigcup X_i \in \sigma(\bigcup \mathcal{C}_i)$. But $\bigcup \mathcal{C}_i$ is a countable union of countable sets, so is countable. Hence $\bigcup X_i \in U$.
   (3) If $X_0 \in U$ then $X_0 \in \sigma(\mathcal{C}_0)$ for some countable $\mathcal{C}_0$. But then $X_0^c \in \sigma(\mathcal{C}_0) \subseteq U$.

2. Show that there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$, but $f(x)$ is *not* of the form $f(x) = \lambda x$. [Hint: Consider $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$ and define a suitable f using a basis for this vector space.]

   Let $\{e_i : i \in I\}$ be a basis for $\mathbb{R}$ as a $\mathbb{Q}$ vector space. Every $x \in \mathbb{R}$ can be written uniquely as a finite sum $\sum_{i \in I_0 \subseteq I} \lambda_i e_i$, $|I_0| < \infty$, $\lambda_i \in \mathbb{Q}$. Pick $i_0 \in I$ and define $f(x) = \lambda_{i_0}$ (or 0 if $i_0 \notin I_0$). Clearly $f(x+y) = f(x) + f(y)$, but $f(e_{i_1}) = 0$ for any $i_1 \neq i_0$. Since $e_{i_1} \neq 0$, if $f(x) = \lambda x$ then $\lambda = 0$. But $f(e_{i_0}) = 1 \neq 0$, a contradiction. Thus f is not of the form $f(x) = \lambda x$ for any $\lambda \in \mathbb{Q}$.

3. (a) If (x_n) and (y_n) are two real sequences, show that

   $$\overline{\lim}\, x_n + \underline{\lim}\, y_n \leq \overline{\lim}(x_n + y_n) \leq \overline{\lim}\, x_n + \overline{\lim}\, y_n$$

   provided both sides are not of the form $\infty - \infty$.

   First note that $\inf_m y_m \leq y_n \leq \sup_m y_m$ for all n.
   Hence $x_n + \inf_m y_m \leq x_n + y_n \leq x_n + \sup_m y_m$ for all n.
   Hence $\sup\{x_n + \inf_m y_m\} \leq \sup(x_n + y_n) \leq \sup\{x_n + \sup_m y_m\}$.
   But $\sup\{x_n + c\} = \sup x_n + c$ unless this is of the form $\infty - \infty$.
   Thus $\sup x_n + \inf y_n \leq \sup(x_n + y_n) \leq \sup x_n + \sup y_n$.
   Now take the limit as $n_0 \to \infty$ of

   $$\sup_{n \geq n_0} x_n + \inf_{n \geq n_0} y_n \leq \sup_{n \geq n_0} (x_n + y_n) \leq \sup_{n \geq n_0} x_n + \sup_{n \geq n_0} y_n$$

   to get

   $$\overline{\lim}\, x_n + \underline{\lim}\, y_n \leq \overline{\lim}(x_n + y_n) \leq \overline{\lim}\, x_n + \overline{\lim}\, y_n$$

(b) Give examples to show that both inequalities may be strict.

Let $x_n = (-1)^n$ and $y_n = -2(-1)^n$. Then

$$\begin{array}{rclcr} \overline{\lim}\, x_n + \underline{\lim}\, y_n &=& 1-2 &=& -1, \\ \overline{\lim}(x_n+y_n) &=& \overline{\lim}\{-(-1)^n\} &=& 1, \\ \overline{\lim}\, x_n + \overline{\lim}\, y_n &=& 1+2 &=& 3. \end{array}$$

4. Let S be a set of positive real numbers. Define

$$\textstyle\sum_{x\in S} x = \sup\left\{\sum_{x\in S_0} x : S_0 \text{ is a finite subset of } S\right\}.$$

Show that if S is uncountable then $\sum_{x\in S} x = +\infty$.

If $x > 0$ then there exists a natural number n such that $xn > 1$, i.e., $x > 1/n$. Let $S_n = \{x \in S : x > 1/n\}$. Then $S = \bigcup_{n=1}^{\infty} S_n$. Since S is uncountable and the union is a countable union, at least one of the S_n must be uncountable. In particular, at least one S_n must be infinite. Letting S_0 be a subset of S_n of size mn, we see $\sum_{x\in S} x \geq mn(1/n) = m$ for any m. Thus $\sum_{x\in S} x = \infty$.

5. Let $(x_n)_{n=0}^{\infty}$ be a real sequence, and for all i, let y_i be a cluster point of $(x_n)_{n=0}^{\infty}$. Show that any cluster point of $(y_i)_{i=0}^{\infty}$ is also a cluster point of $(x_n)_{n=0}^{\infty}$.

Let $L \in \mathbb{R}$ be a cluster point of (y_i). Pick $\varepsilon > 0$ and n_0. Then there exists an $i \geq n_0$ with $|y_i - L| < \varepsilon/2$. Now since y_i is a cluster point of (x_n), there exists an $n \geq n_0$ with $|x_n - y_i| < \varepsilon/2$. But now $|x_n - L| < \varepsilon$. Since this holds for any $\varepsilon > 0$ and n_0, L is a cluster point of (x_n).

Now assume ∞ is a cluster point of (y_i). Pick $K > 0$ and n_0. Then there exists an $i \geq n_0$ with $y_i > 2K$. Now since y_i is a cluster point of (x_n), there exists an $n \geq n_0$ with $|x_n - y_i| < K$ (or $x_n > K$ if $y_i = \infty$). But now $x_n > K$. Since this holds for any $K > 0$ and n_0, ∞ is a cluster point of (x_n). A similar proof holds for $-\infty$.