Math 7350 Homework Assignment 3 Fall 2004

Due October 14.

1. Let $\mathbb{R}(x)$ be the set of *rational functions* $f(x) = \frac{P(x)}{Q(x)}$ where P and Q are polynomials with real coefficients and Q is not identically zero. Define an order $\leq$ by $f \leq g$ if $\exists K > 0 \colon \forall x \geq K \colon f(x) \leq g(x)$.
   (a) Show that with this order, $\mathbb{R}(x)$ is an ordered field. [You may assume it is a field, so you just need to check that $\leq$ is a total order which respects $+$ and $\times$.]
   (b) Show that $\mathbb{R}$ (the set of constant functions) has an upper bound, but no least upper bound in $\mathbb{R}(x)$.

2. Show that the set of (real) cluster points of a sequence (x_n) is a closed set.

3. (a) Show that there are at most 7 sets that can be obtained by applying the operations $^-$ and $^\circ$ iteratively to a set S (including S itself). [Hint: Show that if F is closed, $F^{\circ -} \subseteq F$, and if U is open, $U^{-\circ} \supseteq U$.]
   (b) Give an example where all 7 sets in (a) are distinct.

4. A function $f \colon \mathbb{R} \to \mathbb{R}$ is *upper semi-continuous* at $a \in \mathbb{R}$ if $\forall \varepsilon > 0 \colon \exists \delta > 0 \colon \forall x \colon |x - a| < \delta \Rightarrow f(x) \geq f(a) - \varepsilon$. Show that f is upper semi-continuous for all $x \in \mathbb{R}$ iff $f^{-1}[(a, \infty)]$ is open for all $a \in \mathbb{R}$.

5. A continuous function $g \colon [a, b] \to \mathbb{R}$ is called *piecewise linear* iff there exists a subdivision $a = x_0 < x_1 < \cdots < x_n = b$ such that g is linear on $[x_i, x_{i+1}]$. If $f \colon [a, b] \to \mathbb{R}$ is a continuous function and $\varepsilon > 0$, show that there is a piecewise linear function $g \colon [a, b] \to \mathbb{R}$ such that $|f(x) - g(x)| < \varepsilon$ for all $x \in [a, b]$.

Math 7350 Homework 3 Solutions Fall 2004

1. Let $\mathbb{R}(x)$ be the set of *rational functions* $f(x) = \frac{P(x)}{Q(x)}$ where P and Q are polynomials with real coefficients and Q is not identically zero. Define an order $\leq$ by $f \leq g$ if $\exists K > 0\colon \forall x \geq K\colon f(x) \leq g(x)$.

   (a) Show that with this order, $\mathbb{R}(x)$ is an ordered field. [You may assume it is a field, so you just need to check that $\leq$ is a total order which respects $+$ and $\times$.]

   If $f \leq g$ and $h \leq k$, then $\exists K_1\colon \forall x \geq K_1\colon f(x) \leq g(x)$ and $\exists K_2\colon \forall x \geq K_2\colon h(x) \leq k(x)$. Let $K = \max\{K_1, K_2\}$, then if $x \geq K$, $f(x) + h(x) \leq g(x) + k(x)$, so $f + h \leq g + k$. Thus $\leq$ respects $+$.

   Similarly, if $f \leq g$ and $0 \leq h$, then $\exists K_1\colon \forall x \geq K_1\colon f(x) \leq g(x)$ and $\exists K_2\colon \forall x \geq K_2\colon 0) \leq h(x)$. Let $K = \max\{K_1, K_2\}$, then if $x \geq K$, $f(x)h(x) \leq g(x)h(x)$, so $fh \leq gh$. Thus $\leq$ respects $\times$.

   Similarly, if $f \leq g \leq h$ then $\exists K_1\colon \forall x \geq K_1\colon f(x) \leq g(x)$ and $\exists K_2\colon \forall x \geq K_2\colon g(x) \leq h(x)$. Let $K = \max\{K_1, K_2\}$, then $\forall x \geq K\colon f(x) \leq h(x)$ and so $f \leq h$. Thus $\leq$ is transitive.

   If $f \neq g$, then $h = f - g \neq 0$. Write $h = \frac{P(x)}{Q(x)}$. Then $P \neq 0$. Pick K larger than all the roots of $P(x)$ and $Q(x)$. Then the sign of $P(x)$ and $Q(x)$ does not change for all $x > K$. Hence either $f - g \leq 0$ or $0 \leq f - g$, but not both. Hence either $f \leq g$ or $g \leq f$, but not both. Thus $f \leq g$ and $g \leq f$ imply $f = g$, and for any f and g, either $f \leq g$ or $g \leq f$. Thus $\leq$ is a total order.

   (b) Show that $\mathbb{R}$ (the set of constant functions) has an upper bound, but no least upper bound in $\mathbb{R}(x)$.

   $f(x) = x$ is an upper bound, since for any constant function $g(x) = c$, we can pick $K = c$ and then $\forall x \geq K\colon g(x) = c \leq x = f(x)$. There is no least upper bound, since if f is an upper bound, then for all c, $f \geq c + 1$, so $f - 1 \geq c$. But then $f - 1$ is an upper bound and $f - 1 < f$.

2. Show that the set of (real) cluster points of a sequence (x_n) is a closed set.

   Let $c \in \mathbb{R}$ be a point of closure of the set of cluster points of (x_n). Then for all $m > 0$, there is a cluster point y of (x_n) with $y \in (c - \frac{1}{m}, c + \frac{1}{m})$. Then there are infinitely many n such that $|x_n - y| < \frac{1}{m}$. Inductively define n_m to be the smallest n with $n > n_{m-1}$ and $|x_n - y| < \frac{1}{m}$. Then $|x_{n_m} - c| < \frac{2}{m}$. But then c is a cluster point (limit point) of (x_{n_m}), and so is a cluster point of (x_n).

3. (a) Show that there are at most 7 sets that can be obtained by applying the operations $^-$ and $^\circ$ iteratively to a set S (including S itself). [Hint: Show that if F is closed, $F^{\circ -} \subseteq F$, and if U is open, $U^{-\circ} \supseteq U$.]

   If F is closed, then F is a closed set containing F°. But then F must contain the smallest such set, $F^{\circ -}$. Similarly, if U is open, then U is an open subset of U^-, so must be a subset of the largest such set $U^{-\circ}$.

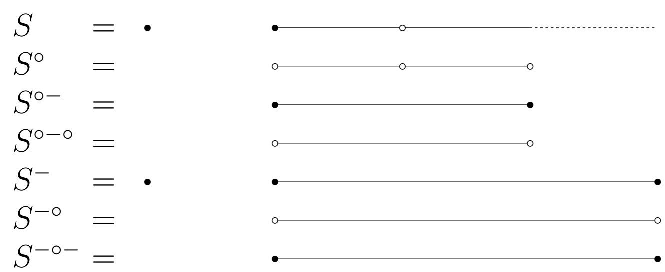
Now let S be arbitrary. Then $S^{\circ -}$ is closed, so $S^{\circ - \circ -} \supseteq S^{\circ -}$. But S° is open, so $S^{\circ - \circ} \subseteq S^{\circ}$. But then $S^{\circ - \circ -} \subseteq S^{\circ -}$. Hence $S^{\circ - \circ -} = S^{\circ -}$. Similarly $S^{- \circ - \circ} = S^{- \circ}$. Since $S^{\circ\circ} = S^{\circ}$ and $S^{--} = S^{-}$, the sets one can obtain are

$$S, \quad S^{\circ}, \quad S^{-}, \quad S^{\circ -}, \quad S^{- \circ}, \quad S^{\circ - \circ}, \quad S^{- \circ -},$$

since applying $^{\circ}$ or $^{-}$ to any of these, again gives one of these sets.

(b) Give an example where all 7 sets in (a) are distinct.

Let $S = \{0\} \cup [1, 2) \cup (2, 3] \cup (\mathbb{Q} \cap [3, 4])$.



4. A function $f \colon \mathbb{R} \to \mathbb{R}$ is *upper semi-continuous* at $a \in \mathbb{R}$ if $\forall \varepsilon > 0 \colon \exists \delta > 0 \colon \forall x \colon |x - a| < \delta \Rightarrow f(x) \geq f(a) - \varepsilon$. Show that f is upper semi-continuous for all $x \in \mathbb{R}$ iff $f^{-1}[(a, \infty)]$ is open for all $a \in \mathbb{R}$.

If f is upper semi-continuous and $x \in f{-}1[(a, \infty)]$ then $f(x) > a$. Set $\varepsilon = f(x) - a$. Then $\exists \delta > 0 \colon |y - x| < \delta \Rightarrow f(y) > f(x) - \varepsilon = a$. Hence $(x - \delta, x + \delta) \subseteq f{-}1[(a, \infty)]$. Thus $f{-}1[(a, \infty)]$ is open.

If $f{-}1[(a, \infty)]$ is open, then for any x and $\varepsilon > 0$, let $a = f(x) - \varepsilon$. Now $x \in f^{-1}[(a, \infty)]$, so $(x - \delta, x + \delta) \subseteq^{-1} [(a, \infty)]$. But then, for all y with $|y - x| < \delta$, $f(y) > a = f(x) - \varepsilon$ and f is upper semi-continuous at x.

5. A continuous function $g \colon [a, b] \to \mathbb{R}$ is called *piecewise linear* iff there exists a subdivision $a = x_0 < x_1 < \cdots < x_n = b$ such that g is linear on $[x_i, x_{i+1}]$. If $f \colon [a, b] \to \mathbb{R}$ is a continuous function and $\varepsilon > 0$, show that there is a piecewise linear function $g \colon [a, b] \to \mathbb{R}$ such that $|f(x) - g(x)| < \varepsilon$ for all $x \in [a, b]$.

We know that f is uniformly continuous, so that for some $\delta > 0$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Choose $a = x_0 < x_1 < \cdots < x_n = b$ so that $|x_{i+1} - x_i| < \delta$, and define g so that $g(x_i) = f(x_i)$ and g is linear on $[x_i, x_{i+1}|$. Then for any $x \in [a, b]$, $x \in [x_i, x_{i+1}]$ for some i. But $g(x) \in [f(x_i), f(x_{i+1})]$ (or $[f(x_{i+1}), f(x_i)]$), so $|f(x) - g(x)| \leq \max\{|f(x) - f(x_i)|, |f(x) - f(x_{i+1})|\} < \varepsilon$ as required.