Math 7350 Homework 4 Solutions Fall 2004

1. Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions. Show that the set of points $C = \{x \in \mathbb{R} : f_n(x) \text{ converges as } n \to \infty\}$ is an $F_{\sigma\delta}$-set.

   The sequence $f_n(x)$ converges iff $f_n(x)$ is a Cauchy sequence: $\forall \varepsilon > 0\colon \exists n_0 > 0\colon \forall n, m \geq n_0\colon |f_n(x) - f_m(x)| < \varepsilon$.
   The '$\sigma\delta$' should take care of the quantifiers, but we need $\{x : |f_n(x) - f_m(x)| < \varepsilon\}$ to be closed. It is not closed as it stands, but the set

   $$E_{n,m,\varepsilon} = \{x : |f_n(x) - f_m(x)| \leq \varepsilon\}$$

   is closed since it is the complement of the set $\{x : |f_n(x) - f_m(x)| > \varepsilon\}$ which is the inverse image of the open set $(-\infty, -\varepsilon) \cup (\varepsilon, \infty)$ under the continuous function $g(x) = f_n(x) - f_m(x)$. Now

   $$C_{n_0,\varepsilon} = \bigcap_{n,m \geq n_0} E_{n,m,\varepsilon} = \{x : \forall n, m \geq n_0\colon |f_n(x) - f_m(x)| \leq \varepsilon\}$$

   is an intersection of closed sets, so is closed. Thus

   $$A_\varepsilon = \bigcup_{n_0} C_{n_0,\varepsilon} = \{x : \exists n_0\colon \forall n, m \geq n_0\colon |f_n(x) - f_m(x)| \leq \varepsilon\}$$

   is an F_σ-set, and

   $$B = \bigcap_k A_{1/k} = \bigcap_{\varepsilon > 0} A_\varepsilon = \{x : f_n(x) \text{ converges}\}$$

   is an $F_{\sigma\delta}$-set.

2. If $E \subset \mathbb{R}$ is a Lebesgue measurable set with a finite measure, prove that for any given $\epsilon > 0$, there is a set U which is a *finite* union of open intervals such that $\lambda(U \triangle E) < \epsilon$. Here $U \triangle E = (U \setminus E) \cup (E \setminus U)$.

   Since $\lambda(E) = \lambda^*(E) < \infty$, we can find a countable sequence of open intervals $I_1, I_2, \dots$ such that $E \subseteq \bigcup_{i=1}^\infty I_i$ and $\sum_{i=1}^\infty \lambda(I_i) = \sum_{i=1}^\infty l(I_i) < \lambda(E) + \varepsilon/2$. If we set $U_\infty = \bigcup_{i=1}^\infty I_i$, then $\lambda(U_\infty) \leq \sum \lambda(I_i) < \lambda(E) + \varepsilon/2$, so $\lambda(U_\infty \setminus E) < \varepsilon/2$.

   Set $U_n = \bigcup_{i=1}^n I_i$. Then $U_1 \subseteq U_2 \subseteq \dots$ and $\bigcup_{n=1}^\infty U_n = U_\infty$. Hence $\lim_{n\to\infty} \lambda(U_n) = \lambda(U_\infty)$. Thus there is some n for which $\lambda(U_n) \geq \lambda(U_\infty) - \varepsilon/2$, and so $\lambda(U_\infty \setminus U_n) < \varepsilon/2$.

   Now U_n is a finite union of intervals and
   $U_n \triangle E = (U_\infty \setminus E) \triangle (U_\infty \setminus U_n) \subseteq (U_\infty \setminus E) \cup (U_\infty \setminus U_n)$,
   so $\lambda(U_n \triangle E) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

3. Show that if U is a *finite* union of open intervals and $\mathbb{Q} \cap [0, 1] \subseteq U$, then $\lambda(U) \geq 1$.

   Suppose $U = \bigcup_{i=1}^n I_i$ where I_i are open intervals and $\mathbb{Q} \cap [0, 1] \subseteq U$. We need to show $\lambda(U) \geq 1$.

   Without loss of generality, and by induction of n, we may assume the I_i's are pairwise disjoint. Otherwise, if $I_i \cap I_j \neq \emptyset$, then $I_i \cup I_j$ is also an open interval, and we can write U as a union of a smaller number of intervals. Similarly, without loss of generality, each I_i intersects $[0,1]$, otherwise we could remove I_i from the union to get a set U' which is the union of $n-1$ intervals, $\mathbb{Q}\cap[0,1]\subseteq U'$, and $\lambda(U)\geq\lambda(U')\geq 1$.

Now write the intervals as $I_i=(a_i,b_i)$ and order the intervals so that $a_1<a_2<\cdots<a_n$ (if $a_i=a_j$ then I_i and I_j intersect, so we may assume the a_i are distinct). Now if $b_i>a_{i+1}$ then I_i and I_{i+1} intersect. Hence we may assume $b_i\leq a_{i+1}$. In particular we may assume $U\cap(a_i,a_{i+1})=I_i$. On the other hand, if $b_i<a_{i+1}$ then either $b_i\geq 1$ (so $I_{i+1}\cap[0,1]=\emptyset$) or $a_{i+1}\leq 0$ (so $I_i\cap[0,1]=\emptyset$) or $(b_i,a_{i+1})\cap[0,1]\neq\emptyset$ (so $(b_i,a_{i+1})\cap[0,1]$ contains a rational that is not in U). Since none of these are possible, we must have $b_i=a_{i+1}$ for all i with $1\leq i<n$. Thus $U=(a_1,b_n)\setminus\{a_2,\dots,a_n\}$. But $0\in U$ so $a_1<0$, and $1\in U$ so $b_n>1$. Now $\lambda(U)=b_n-a_1>1$.

4. The first Borel-Cantelli Lemma states that if the sets $B_1,B_2,\dots$ are measurable and $\sum_{i=1}^{\infty}\lambda(B_i)<\infty$, then the set of points that belong to infinitely many B_i is a set of measure 0.

   Prove the first Borel-Cantelli Lemma.

   Let E be the set of points that belong to infinitely many B_i. Since $\sum_{i=1}^{\infty}\lambda(B_i)$ converges, for any $\varepsilon>0$ there is an n_0 such that $\sum_{i=n_0}^{\infty}\lambda(B_i)<\varepsilon$. But then $\lambda(\bigcup_{i=n_0}^{\infty}B_i)\leq\sum_{i=n_0}^{\infty}\lambda(B_i)<\varepsilon$. Now if $x\in E$ then x is in infinitely many B_i, and so lies in some B_i with $i\geq n_0$. Thus $E\subseteq\bigcup_{i=n_0}^{\infty}B_i$. Hence $\lambda(E)\leq\lambda(\bigcup_{i=n_0}^{\infty}B_i)<\varepsilon$. Since this holds for all $\varepsilon>0$, $\lambda(E)=0$.

5. Let A be a subset of $\mathbb{R}$ such that $\lambda(A)>0$. Denote by $A-A$ the set $\{x-y : x,y\in A\}$.

   (a) Prove that there is an interval $[a,b]$ such that $\lambda(A\cap[a,b])>\frac{3}{4}(b-a)$.
   First by setting $A_n=A\cap[n,n+1)$ and noting that and $\sum\lambda(A_n)=\lambda(A)$, we may assume $\lambda(A_n)>0$ for some n. Replacing A by A_n, we may assume $\lambda(A)<\infty$. Now $A\subseteq U$ with U open and $\lambda(U)<\frac{4}{3}\lambda(A)$. Writing U as a countable disjoint union of intervals I_i, we get $\sum\lambda(I_i)=\lambda(U)<\frac{4}{3}\lambda(A)=\frac{4}{3}\sum\lambda(I_i\cap A)$. If $\lambda(I_i)\geq\frac{4}{3}\lambda(I_i\cap A)$ for all i we obtain a contradiction. Hence there is an interval I_i with $\lambda(I_i\cap A)>\frac{3}{4}\lambda(I_i)$.

   (b) Show that if $0\leq\delta\leq\frac{1}{4}(b-a)$ then $A\cap(A+\delta)\cap[a,b]$ is non-empty.
   Let $A'=A\cap[a,b]$. Now if $A'\cap(A'+\delta)=\emptyset$ then $\lambda(A')+\lambda(A'+\delta)=\lambda(A'\cup(A'+\delta))\leq\lambda([a,b+\delta])\leq\frac{5}{4}(b-a)$. But $\lambda(A')=\lambda(A'+\delta)>\frac{3}{4}(b-a)$, a contradiction. Hence $A'\cap(A'+\delta)\subseteq A\cap(A+\delta)\cap[a,b]$ is non-empty.

   (c) Deduce that $A-A\supseteq[-\frac{1}{4}(b-a),\frac{1}{4}(b-a)]$.
   If $0\leq\delta\leq\frac{1}{4}(b-a)$ then $A\cap(A+\delta)\neq\emptyset$, so $x\in A\cap(A+\delta)$ for some x. But then $x\in A$ and $x-\delta\in A$, so $\pm x\in A-A$. Thus $A-A\supseteq[-\frac{1}{4}(b-a),\frac{1}{4}(b-a)]$.