Math 7350 Homework 5 Solutions Fall 2004

1. Evaluate, with justification, the limit $\lim_{n\to\infty}\int_0^\infty (1+x/n)^{-n}\,n\sin(x/n)\,dx$.

   $\log(1+\frac{x}{n})^{-n} = -n(\frac{x}{n}+O((\frac{x}{n})^2)) = -x+O(\frac{x^2}{n}) \to -x$ as $n\to\infty$, thus $(1+\frac{x}{n})^{-n}\to e^{-x}$ as $n\to\infty$. Also $n\sin(\frac{x}{n}) = n(\frac{x}{n}+O((\frac{x}{n})^3)) = x+O(\frac{x^3}{n^2}) \to x$ as $n\to\infty$. Thus if we set $f_n(x) = (1+x/n)^{-n}\,n\sin(x/n)$ then $f_n\to f$ as $n\to\infty$ where $f(x) = xe^{-x}$. We would expect the integral to converge to $\int_0^\infty xe^{-x}\,dx$. To prove this, use Dominated Convergence Theorem. Since $\sin x\le x$ for $x\ge 0$ we have $|f_n(x)|\le x(1+x/n)^{-n}$. If we assume $n\ge 3$ then

   $(1+\frac{x}{n})^n \ge 1+\binom{n}{1}\frac{x}{n}+\binom{n}{2}\frac{x^2}{n^2}+\binom{n}{3}\frac{x^3}{n^3} \ge 1+x+\frac{1}{2}(1-\frac{1}{n})x^2+\frac{1}{6}(1-\frac{1}{n})(1-\frac{2}{n})x^3 \ge x+\frac{1}{27}x^3.$

   Thus $|f_n(x)|\le g(x) = 1/(1+\frac{x^2}{27})$, which is integrable. Thus the limit is $\int_0^\infty xe^{-x}\,dx$. Now letting $g_n = xe^{-x}\chi_{[0,n]}$, $\int g_n\to\int f$ by MCT. But $\int_0^n xe^{-x}\,dx = [-(x+1)e^{-x}]_0^n = 1-(n+1)e^{-n}\to 1$ as $n\to\infty$. Thus the limit is 1.

2. Let S be a measurable set with finite measure. Show that $f(x) = \lambda(S\cap(S+x))$ is a continuous function and $\lim_{x\to+\infty} f(x) = 0$.

   If $I = (a,b)$ is an interval and $z\in S\cap(I+x)$ but $z\notin S\cap(I+y)$ then $z\in(I+x)\setminus(I+y)$. Now $\lambda((I+x)\setminus(I+y)) = \lambda((a+x,b+x)\setminus(a+y,b+y)) \le |x-y|$. Thus $\lambda(S\cap(I+x))-\lambda(S\cap(I+y))\le|x-y|$. If $U=\bigcup_{i=1}^N I_i$ is a disjoint union of N intervals, then $\lambda(S\cap(U+x)) = \sum\lambda(S\cap(I_i+x))$. Thus $\lambda(S\cap(U+x))-\lambda(S\cap(U+y))\le N|x-y|$. If $\lambda(S)<\infty$, then there is such a U with $\lambda(S\triangle U)<\varepsilon$. Then $|\lambda(S\cap(S+x))-\lambda(S\cap(U+x))|<\varepsilon$. Thus $f(x)-f(y) = \lambda(S\cap(S+x))-\lambda(S\cap(S+y)) < 2\varepsilon+N|x-y|$. Similarly $f(y)-f(x)<2\varepsilon+N|y-x|$, so $|f(y)-f(x)|<2\varepsilon+N|x-y|$. Since this is $<3\varepsilon$ for $|x-y|$ sufficiently small, and $\varepsilon>0$ is arbitrary, $f(x)$ is continuous.

   Let K be the maximum distance between any two points in the (bounded) set U. Let $V = S\setminus U$. Since $S\subseteq U\cup V$, $S\cap(S+x)\subseteq(U\cap(U+x))\cup(U\cap(V+x))\cup(V\cap(S+x))$. If $x>K$ then $U\cap(U+x)=\emptyset$, so $\lambda(S\cap(S+x))\le\lambda(V+x)+\lambda(V)<2\varepsilon$. Hence $f(x)\to 0$ as $x\to+\infty$.

3. Suppose $f\colon[0,1]\to\mathbb{R}$ and f^n is integrable for all $n\ge 1$. If $\int_0^1 f^n = \int_0^1 f$ for all $n\ge 1$, show that $f=\chi_S$ a.e. for some measurable set $S\subseteq[0,1]$.

   We start be showing $f^2 = f$ a.e.. Consider $g = (f^2-f)^2$. Then $g\ge 0$ and $\int_0^1 g = \int_0^1(f^4-2f^3+f^2) = \int_0^1 f - 2\int_0^1 f + \int_0^1 f = 0$. Let $E_k = \{x : g(x)>1/k\}$. Then $0 = \int g \ge \lambda(E_k)/k$. Thus $\lambda(E_k)=0$. But then $\lambda(\bigcup E_k) = 0$ and $\bigcup E_k = \{x : g(x)>0\} = \{x : f(x)^2\ne f(x)\}$. Thus $f^2 = f$ a.e.. Since f is measurable, $S = f^{-1}[\{1\}]$ is measurable. If $f(x)\ne\chi_S(x)$ then $f(x)\ne 0,1$, so $f(x)^2\ne f(x)$. Thus $f=\chi_S$ a.e..

4. (a) Show that if S is measurable with finite measure then $\lim_{n\to\infty}\int_S \cos nx\,dx = 0$. [Hint: Recall that there exists a finite union U of intervals with $\lambda(S \triangle U) < \varepsilon$.]

Assume first that $S = (a, b)$ is an open interval. Then

$$|\int_S \cos nx\,dx| = |\int_a^b \cos nx\,dx| = |[\tfrac{1}{n}\sin(nx)]_a^b| \le \tfrac{2}{n}.$$

Fix $\varepsilon > 0$ and let U be a union of k open intervals with $\lambda(S \triangle U) < \varepsilon$. Now

$|\int_S \cos nx\,dx - \int_U \cos nx\,dx| = |\int(\chi_S - \chi_U)\cos nx\,dx| \le \int|\chi_S - \chi_U| = \lambda(S \triangle U)$.

Thus $|\int_S \cos nx\,dx - \int_U \cos nx\,dx| < \varepsilon$ and $|\int_U \cos nx\,dx| \le k(2/n)$. Hence $|\int_S \cos nx\,dx| < \varepsilon + k(2/n)$ which is $< 2\varepsilon$ for sufficiently large n. Since this holds for any $\varepsilon > 0$, $\lim_{n\to\infty}\int_S \cos nx\,dx = 0$.

(b) Deduce the Riemann-Lebesgue Theorem:

If f is integrable then $\lim_{n\to\infty}\int f(x)\cos nx\,dx = 0$.

By writing $f = f_+ - f_-$ it is enough to prove the result when $f \ge 0$ and $\int f < \infty$. Let ϕ be a simple function with $0 \le \phi \le f$, $\phi = \sum_{i=1}^k a_i\chi_{S_i}$, and $\int\phi \ge \int f - \varepsilon/2$. Then $\int\phi\cos nx\,dx = \sum_{i=1}^k a_i\int_{S_i}\cos nx\,dx \to 0$ as $n \to \infty$. Thus for sufficiently large n, $|\int\phi\cos nx\,dx| < \varepsilon/2$. But $|\int(f(x) - \phi(x))\cos nx\,dx| \le \int(f - \phi) \le \varepsilon/2$, so $|\int f(x)\cos nx\,dx| \le |\int\phi(x)\cos nx\,dx| + |\int(f - \phi)\cos nx\,dx| < \varepsilon$ for sufficiently large n.

5. Suppose that for all n, $\int_0^1 f_n(x)^2\,dx \le \frac{1}{n^4}$. Show that $f_n \to 0$ a.e. on $[0, 1]$.

Let $E_n = \{x : |f_n(x)| > 1/n\}$. Then $\int_0^1 f_n(x)^2\,dx \ge \lambda(E_n)/n^2$. Thus $\lambda(E_n) \le 1/n^2$. If $f_n(x) \not\to 0$ then there is some $\varepsilon = 1/k$ with $f_n(x) > 1/k$ for infinitely many n. But then $f_n(x) > 1/n$ for infinitely many n. Thus $x \in \bigcup_{n_0}^\infty E_k$. But the measure of this set is at most $\sum_{n_0}^\infty 1/n^2 \to 0$ as $n_0 \to \infty$. Thus the set of x where $f_n \not\to 0$ is of measure zero.

Alternative solution:

Let $g(x) = \sum_{n=1}^\infty f_n(x)^2 \in [0, \infty]$. Now $\sum_{n=1}^N f_n^2$ is increasing in N, so by MCT,

$$\int g = \int \lim_N \sum_{n=1}^N f_n^2 = \lim_N \int \sum_{n=1}^N f_n^2 = \lim_N \sum_{n=1}^N \int f_n^2 \le \sum_{n=1}^\infty \tfrac{1}{n^4} < \infty.$$

But this implies $\{x : g(x) = +\infty\}$ has measure 0. Thus $g(x) = \sum_{n=1}^\infty f_n(x)^2$ converges a.e., but this implies $f_n(x)^2 \to 0$ a.e., and so $f_n(x) \to 0$ a.e..