Math 7350 Homework 6 Solutions Fall 2004

1. Suppose that f is continuous on $(0,\infty)$ and $\int_0^\infty e^{-ax}|f(x)|\,dx \le 1$ for all $a>0$. Prove that f is integrable on $(0,\infty)$ and that

$$\int_0^\infty f(x)\,dx = \lim_{n\to\infty}\int_0^\infty e^{-x/n}f(x)\,dx.$$

Let $f_n(x)=e^{-x/n}f(x)$. Then $|f_n(x)|=e^{-x/n}|f(x)|$ is an increasing sequence of non-negative measurable functions and $\lim_{n\to\infty}|f_n(x)|=|f(x)|$. Now by MCT $\int|f| = \lim\int|f_n|$. But $\int|f_n|\le 1$, so $\int|f|\le 1$ and hence f is integrable. Since $|f_n|\le|f|$ and $|f|$ is integrable, by DCT, $\int f=\lim\int f_n$.

2. A function $f\colon [a,b]\to\mathbb{R}$ is called *singular* if $f'=0$ a.e.. Show that any increasing function f is the sum of an increasing absolutely continuous function and an increasing singular function. [Hint: $\int f'$.]

Since f is increasing, we know f' exists a.e., $f'\ge 0$ a.e., and $\int_a^b f'(t)\,dt \le f(b)-f(a)$. Let $g(x)=\int_a^x f'(t)\,dt$. Then $g\colon[a,b]\to\mathbb{R}$ is absolutely continuous, (since it is the integral of an integrable function) and increasing (since $f'\ge 0$). Let $h(x)=f(x)-g(x)$. Then $f(x)=g(x)+h(x)$, so it is enough to show that $h'=0$ a.e., and h is increasing. Now $h'=f'-g'$ exists a.e., since f,g are increasing. Also, by FTC $g'(x)=f'(x)$ a.e., so $h'=0$ a.e.. If $x<y$ then $g(y)-g(x)=\int_x^y f' \le f(y)-f(x)$, so $h(y)\ge h(x)$ and h is increasing.

3. Suppose $f\colon (a,b)\to\mathbb{R}$ is differentiable everywhere in (a,b) and $[c,d]\subseteq(a,b)$.

   (a) Show that if f' is continuous on $[c,d]$ then f is absolutely continuous on $[c,d]$.

   If f' is continuous on $[c,d]$ then f' is bounded on $[c,d]$. Say $|f'|\le M$. Consider disjoint intervals $I_i=(a_i,b_i)$, $i=1,\dots,n$ with $\sum_{i=1}^n(b_i-a_i)<\varepsilon/M$. Then by the Mean Value Theorem, $f(b_i)-f(a_i)=(b_i-a_i)f'(c_i)$ for some $c_i\in I_i$, so $|f(b_i)-f(a_i)|\le M(b_i-a_i)$. Now $\sum_{i=1}^n|f(b_i)-f(a_i)|\le\sum_{i=1}^n M(b_i-a_i) < M(\varepsilon/M)=\varepsilon$. Hence f is absolutely continuous.

   (b) Give an example of such an f which not absolutely continuous on $[c,d]$. [Hint: Consider $f(x)=h(x)\cos(1/x)$ on $[-1,1]$ which is not of bounded variation, but $f'(0)$ exists.]

   Let $h(x)=x/\log(2/|x|)$ when $x\ne 0$ and $h(x)=0$. For $0<|x|<2$, $f(x)=h(x)\cos(1/x)$ is differentiable. Also, $h'(0)=\lim_{\varepsilon\to 0}\varepsilon^{-1}h(\varepsilon)\cos(1/\varepsilon) = \lim_{\varepsilon\to 0}(1/\log(2/|\varepsilon|))\cos(1/\varepsilon)=0$, so $f'(0)=0$ exists as well. Thus f' exists in $[-1,1]$. Consider the points $a_k=1/(\pi k)$. Then $f(a_k)=\frac{(-1)^k}{\pi k\log(2\pi k)}$. Now $|f(a_k)-f(a_{k-1})|\ge\frac{2}{\pi k\log(2\pi k)}$ and so $\sum_{k=2}^N|f(a_k)-f(a_{k-1})|\ge\sum_{k=2}^N\frac{0.1}{k\log k}$. But $\sum\frac{1}{k\log k}$ diverges (e.g., by integral test). Thus f is not of bounded variation, and so cannot be absolutely continuous.

   Alternative function: $x^2\cos(1/x^2)$ works as well.

4. Give an example of a function $f\colon [0,1] \to \mathbb{R}$ that is absolutely continuous and strictly increasing, but $f' = 0$ on a set of strictly positive measure.
[Hint: Consider the integral of χ_S where S is some suitable set.]

Enumerate the rationals in $(0,1)$ as $q_1, q_2, \dots$. Fix $\varepsilon > 0$ and consider the set $S = \bigcup_{k=1}^{\infty}(q_k - \varepsilon/2^k, q_k + \varepsilon/2^k)$. Then $\lambda(S) \le \sum 2\varepsilon/2^k = 2\varepsilon$. Setting $\varepsilon = 1/4$, we have $\lambda(S) < 1$. Let $f(x) = \int_0^x \chi_S(t)\,dt$. Then f is absolutely continuous and $f' = \chi_S$ a.e.. But $\chi_S(t) = 0$ on a set $[0,1] \setminus S$ of positive measure. Also, if $0 \le x < y \le 1$ then $f(y) - f(x) = \int_x^y \chi_S(t)\,dt = \lambda(S \cap [x,y])$. Pick a rational $q_k \in (x,y)$. Then $\lambda(S \cap [x,y]) \ge \lambda((q_k - \varepsilon/2^k, q_k + \varepsilon/2^k) \cap [x,y]) \ge \min(|y-x|, \varepsilon/2^k) > 0$. Thus f is strictly increasing.

5. Suppose $g\colon [a,b] \to [c,d]$ is an increasing function with $g(a) = c$ and $g(b) = d$. Suppose also that g is absolutely continuous.

   (a) Show that if $S \subseteq [c,d]$ is measurable then $\lambda(S) = \int_a^b \chi_S(g(t))g'(t)\,dt$.
   [Hint: Consider intervals first.]

   Define $\mu(S) = \int_a^b \chi_S(g(t))g'(t)\,dt$. First consider any interval $I \subseteq [c,d]$ with endpoints c_i and d_i, say, (may be open or closed). Since g is increasing, $g^{-1}[I]$ is an interval, with endpoints a_i and b_i, say. Since g is continuous, $g(a_i) = c_i$ and $g(b_i) = d_i$. Now

   $$\mu(I) = \int_a^b \chi_I(g(t))g'(t)\,dt = \int_{a_i}^{b_i} g'(t) = g(b_i) - g(a_i) = d_i - c_i = \lambda(I).$$

   By linearity, $\mu(U) = \lambda(U)$ for any finite disjoint union of open intervals. Now let $U = \bigcup_{i=1}^{\infty} I_i$, I_i disjoint open intervals. Since $g' \ge 0$, by MCT

   $$\mu(U) = \int \sum \chi_{I_i}(g(t))g'(t)\,dt = \sum \int \chi_{I_i}(g(t))g'(t)\,dt = \sum \lambda(I_i) = \lambda(U).$$

   Thus $\mu(U) = \lambda(U)$ for all open $U \subseteq [c,d]$. Now, by linearity, $\mu([c,d] \setminus U) = \mu([c,d]) - \mu(U) = \lambda([c,d]) - \lambda(U) = \lambda([c,d] \setminus U)$, so $\mu(F) = \lambda(F)$ for all closed sets F as well. Now for any measurable S we can find open U and closed F with $F \subseteq S \subseteq U$ and $\lambda(U) < \lambda(F) + \varepsilon$. Now $\lambda(F) = \mu(F) \le \mu(S) \le \mu(U) = \lambda(U) < \lambda(F) + \varepsilon$. Letting $\varepsilon \to 0$ we get $\mu(S) = \lambda(S)$ for all measurable $S \subseteq [c,d]$.

   (b) Deduce that if $f\colon [c,d] \to \mathbb{R}^*$ is integrable then $\int_c^d f(t)\,dt = \int_a^b f(g(t))g'(t)\,dt$.

   By writing $f = f_+ - f_-$, $f_+, f_- \ge 0$, we may assume $f \ge 0$. Let ϕ_n be an increasing sequence of simple functions tending pointwise to f. Now $\int_c^d \chi_S(t)\,dt = \int_a^b \chi_S(g(t))g'(t)\,dt$, so by linearity $\int_c^d \phi_n(t)\,dt = \int_a^b \phi_n(g(t))g'(t)\,dt$. Thus by MCT (twice, using $g' \ge 0$)

   $$\int_c^d f(t)\,dt = \lim \int_c^d \phi_n(t)\,dt = \lim \int_a^b \phi_n(g(t))g'(t)\,dt = \int_a^b f(g(t))g'(t)\,dt.$$