Math 7350 Homework 7 Solutions Fall 2004

1. Show that $\displaystyle\int_0^\infty \frac{e^{-x}}{1+x}\,dx \le \sqrt[3]{\tfrac{2}{9}}$.

   Let $f(x) = \frac{1}{1+x}$ and $g(x) = e^{-x}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then by Hölder,

   $$\int_0^\infty \frac{e^{-x}}{1+x}\,dx \le \left(\int_0^\infty \frac{1}{(1+x)^p}\,dx\right)^{1/p} \left(\int_0^\infty e^{-qx}\,dx\right)^{1/q} = \left(\tfrac{1}{p-1}\right)^{1/p} \left(\tfrac{1}{q}\right)^{1/q}.$$

   Take $p = 3$ and $q = \frac{3}{2}$, then $\frac{1}{p} + \frac{1}{q} = 1$ and $\left(\frac{1}{p-1}\right)^{1/p}\left(\frac{1}{q}\right)^{1/q} = \sqrt[3]{\frac{2}{9}}$.

2. Let $C[0,1]$ be the space of all continuous functions on $[0,1]$.

   (a) Show that with the $\|\cdot\|_\infty$ norm, $C[0,1]$ is a Banach space.

   If $f_n \in C[0,1]$ and f_n is Cauchy in L^∞, then $\forall \varepsilon > 0\colon \exists n_0\colon \forall n, m \ge n_0\colon$ $\|f_n - f_m\|_\infty < \varepsilon$. But $\|f_n - f_m\|_\infty < \varepsilon$ implies $|f_n(x) - f_m(x)| < \varepsilon$ for a.e. x. Since $f_n - f_m$ is continuous, we must have $|f_n(x) - f_m(x)| \le \varepsilon$ for all x, so $f_n(x)$ is a Cauchy sequence. Let $f(x) = \lim f_n(x)$. Then $|f_n(x) - f(x)| \le \varepsilon$ for all $n \ge n_0$. Since n_0 is independent of x, $f_n \to f$ uniformly in x. Thus $\|f_n - f\|_\infty \to 0$ and $f_n \to f$ in L^∞. A uniform limit of continuous functions is continuous, so $f \in C[0,1]$. Thus $C[0,1]$ is a complete normed space with the L^∞-norm. Thus it is a Banach space.

   (b) Show that with the $\|\cdot\|_p$ norm, $C[0,1]$ is not a Banach space for $1 \le p < \infty$.

   Let $f_n = \frac{(2x)^n}{1+(2x)^n}$. Then $\lim_{n\to\infty} f_n(x) = f(x)$ where $f(x) = 0$ for $x < 1/2$, $f(x) = 1$ for $x > 1/2$ and $f(1/2) = 1/2$. Since $|f_n - f|^p \le 1$, by DCT, $\int |f_n - f|^p \to 0$, so $\|f_n - f\|_p \to 0$, and $f_n \to f$ in $L^p([0,1])$. Hence f_n is a Cauchy sequence in $C[0,1]$ with the L^p-norm. If $f_n \to g$ in $C[0,1]$ with the L^p-norm, then $f_n \to g$ in $L^p([0,1])$. But then $g = f$ a.e.., but f is not $=$ a.e., to any continuous function.

3. (a) If $f\colon [0,1] \to \mathbb{R}$ is measurable, show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

   Suppose $\|f\|_\infty > M$. Then $S = \{x : |f| > M\}$ has positive measure and $\|f\|_p \ge (\lambda(S)M^p)^{1/p} = M(\lambda(S))^{1/p}$. Thus $\underline{\lim}_{p\to\infty} \|f\|_p \ge \lim_{p\to\infty} M(\lambda(S))^{1/p} = M$. Since this holds for all $M < \|f\|_\infty$, $\underline{\lim}_{p\to\infty} \|f\|_p \ge \|f\|_\infty$.

   Conversely, $\|f\|_\infty \le M$. Then $|f| \le M$ a.e., so $\|f\|_p \le (M^p)^{1/p} = M$. Thus $\overline{\lim}_{p\to\infty} \|f\|_p \le M$. Since this holds for all $M \ge \|f\|_\infty$, $\overline{\lim}_{p\to\infty} \|f\|_p \le \|f\|_\infty$.

   Thus $\lim_{p\to\infty} \|f\|_p$ exists and equals $\|f\|_\infty$.

   (b) Give an example to show that this statement may be false if f is defined on the whole of $\mathbb{R}$.

   Let $f(x) = 1$ on $\mathbb{R}$. Then $\|f\|_p = \infty$ for all $p < \infty$ and $\|f\|_\infty = 1$.

4. Suppose $g \in L^p(\mathbb{R})$, $|f_n| \leq M$, and $f_n \to f$ a.e.. Show that $f_n g \to fg$ in $L^p(\mathbb{R})$.

   We need to show that $\|f_n g - fg\|_p \to 0$ as $n \to \infty$, or equivalently, we need to show $\lim_{n\to\infty} \int |f_n g - fg|^p = 0$. Now $\int |g|^p < \infty$ and $|f_n - f| \leq 2M$, so $|f_n g - fg|^p \leq (2M)^p |g|^p$, which is integrable. Thus by DCT, $\lim_{n\to\infty} \int |f_n g - fg|^p = \int \lim_{n\to\infty} |f_n g - fg|^p = \int 0 = 0$.

5. Assume $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f_n \in L^p(\mathbb{R})$ with $\|f_n\|_p \leq M$ for all n and suppose $f_n \to 0$ a.e..

   (a) If $\lambda(S) < \infty$, show that $\int_S f_n \to 0$.

   Fix $\varepsilon > 0$ and let $S_{n_0} = \{x \in S : \forall n \geq n_0 \colon |f_n(x)| < \varepsilon\}$. Then S_{n_0} is an increasing sequence of sets with $\lambda(S \setminus \bigcup S_{n_0}) = 0$. In particular, there exists an n_0 with $\lambda(S \setminus S_{n_0}) < \varepsilon$. Now, for $n \geq n_0$, $|\int_{S_{n_0}} f_n| \leq \lambda(S)\varepsilon$ since $|f_n| \leq \varepsilon$ on $S_{n_0} \subseteq S$, and $|\int_{S \setminus S_{n_0}} f_n| \leq (\int_{S\setminus S_{n_0}} 1^q)^{1/q} (\int_{S \setminus S_{n_0}} |f_n|^p)^{1/p} \leq \varepsilon^{1/q} M$ by Hölder. Thus $|\int_S f_n| \leq |\int_{S_{n_0}} f_n| + |\int_{S\setminus S_{n_0}} 1 f_n| \leq \varepsilon\lambda(S) + \varepsilon^{1/q} M$ for all $n \geq n_0$. Since ε was arbitrary, the result follows.

   (b) Using (a), show that for all $g \in L^q(\mathbb{R})$, $\int f_n g \to 0$.

   By (a) this is true when $g = \chi_S$ and $\lambda(S) < \infty$. Thus by linearity it is true for all simple functions. Let g_0 be a simple function with $|g_0| \leq |g|$ and $\|g - g_0\|_q \leq \varepsilon$ (L^q functions can be approximated by simple functions for $q < \infty$). Then $|\int f_n g| \leq |\int f_n g_0| + |\int f_n (g - g_0)| \leq |\int f_n g_0| + M\varepsilon$ by Hölder. For sufficiently large n, $|\int f_n g_0| \leq M\varepsilon$, so $|\int f_n g| \leq 2M\varepsilon$. Since ε is arbitrary, $\int f_n g \to 0$.

   (c) Does (b) hold in the case $p = 1$, $q = \infty$?

   No. For example, take $g = 1$, $f_n = \chi_{[n,n+1]}$.