Math 7411 Homework Assignment 2 Spring 2008

Due Tuesday, February 5.

1. Assume $f\colon X \to Y$ is a function. Prove the following.
   (a) f is injective iff for all Z and all maps $g, h\colon Z \to X$, $f \circ g = f \circ h \Rightarrow g = h$.
   (b) f is surjective iff for all Z and all maps $g, h\colon Y \to Z$, $g \circ f = h \circ f \Rightarrow g = h$.

2. Assume A and B are well ordered sets, with well orderings $\leq$ and $\leq'$ respectively. If A is order isomorphic to an initial segment B' of B and B is order isomorphic to an initial segment A' of A, show that $A' = A$ and $B' = B$. In particular, deduce that A and B are order isomorphic. [Hint: show that the composition of the two order isomorphisms is the identity.]

   Let X be a set. Let $\mathcal{X}$ be the collection of pairs $(A, \leq)$ where $\leq$ is a well ordering of A and $A \subseteq X$. Define an equivalence relation on $\mathcal{X}$ by $(A, \leq) \sim (B, \leq')$ if $(A, \leq)$ is order-isomorphic to $(B, \leq')$. Define an ordering $\preccurlyeq$ on $\bar{\mathcal{X}} = \mathcal{X}/{\sim}$ by $\overline{(A, \leq)} \preccurlyeq \overline{(B, \leq')}$ if $(A, \leq)$ is order isomorphic to an initial segment of $(B, \leq')$. Note that $\preccurlyeq$ is well defined, and, by Lemma 1 of the notes, $\preccurlyeq$ is a total ordering.

3. Show that if $(A, \leq) \in \mathcal{X}$ then $(A, \leq)$ is order isomorphic to the initial segment $\{\overline{(B, \leq')} \in \bar{\mathcal{X}} \mid \overline{(B, \leq')} \prec \overline{(A, \leq)}\}$. Deduce that $(\bar{\mathcal{X}}, \preccurlyeq)$ is well ordered.

4. Suppose that X has a well ordering. Show without using AC that there exists a well ordered set Y with $|Y| > |X|$. [Hint: if there was a injection from $Y = \bar{\mathcal{X}}$ to X in question 3, then there would exist an $(A, \leq) \in \mathcal{X}$ that was order isomorphic both to $(\bar{\mathcal{X}}, \preccurlyeq)$ and to a proper initial segment of $(\bar{\mathcal{X}}, \preccurlyeq)$, contradicting question 2.]

5. We call a set X *Dedekind finite* if there is no bijection $f\colon X \to X'$ between X and a proper subset X' of X.
   (a) Show that if X is not Dedekind finite, then there is an injection $g\colon \mathbb{N} \to X$. [Hint: If $x_0 \notin X'$ and $x_{n+1} = f(x_n)$ then the x_n are distinct.]
   (b) Show that if there exists an injection $g\colon \mathbb{N} \to X$ then X is not Dedekind finite.