1. Suppose $A$, $B$, and $C$ are mutually independent events with $\mathbb{P}(A) = 0.5$, $\mathbb{P}(B) = 0.3$, and $\mathbb{P}(C) = 0.2$. Calculate

(a) $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = 0.5 \times 0.3 \times 0.2 = 0.03$.

(b) $\mathbb{P}(A \cup B \cup C) = 1 - \mathbb{P}(A' \cap B' \cap C') = 1 - 0.5 \times 0.7 \times 0.8 = 1 - 0.28 = 0.72$.

   (One can also use inclusion-exclusion.)

(c) $\mathbb{P}(B \mid C) = \mathbb{P}(B) = 0.3$ as $B$ is independent of $C$.

2. Suppose now that $A$, $B$, and $C$ are mutually exclusive events with $\mathbb{P}(A) = 0.5$, $\mathbb{P}(B) = 0.3$, and $\mathbb{P}(C) = 0.2$. Calculate

(a) $\mathbb{P}(A \setminus B \setminus C) = \mathbb{P}(A)$, as no two can occur at the same time.

(b) $\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) = 0.5 + 0.3 + 0.2 = 1$.

(c) $\mathbb{P}(B \mid C) = \mathbb{P}(B \setminus C) = \mathbb{P}(C) = 0$.

3. How many anagrams are there of the word “bookkeeper”?

There are 3 e’s, 2 k’s, 2 o’s, and 1 each of b, p, r. The number of anagrams is the multinomial coefficient \( \binom{10}{3,2,2,1,1,1} = \frac{10!}{3!2!2!1!1!1!} = 151200 \).

4. Five cards are taken from a standard deck of 52 cards. What is the probability of the following events.

(a) All cards are from the same suit. 4 choices of suit. Then choose 5 cards from 13. Total sample space is choosing 5 cards from 52. Thus probability is $4 \left(\binom{13}{5}\right) / \left(\binom{52}{5}\right) = \frac{5148}{2598960} \approx 0.00198$.

(b) Among the five cards, every suit occurs at least once. One suit will have 2 cards the others only 1. There are 4 choices for the suit containing two cards. Probability is $4 \left(\binom{13}{2}\right) \left(\binom{13}{1}\right) \left(\binom{13}{1}\right) / \left(\binom{52}{5}\right) = \frac{685464}{2598960} \approx 0.2637$.

5. Suppose a game is played, rolling a fair six-sided die, with the following pay-offs:

Roll 1,2,3 $\Rightarrow X = -1$; Roll 4 or 5 $\Rightarrow X = +1$; Roll a 6 $\Rightarrow X = +2$.

(a) Calculate the mean $\mathbb{E}(X)$.

$\mathbb{E}(X) = -1 \times \frac{3}{6} + 1 \times \frac{3}{6} + 2 \times \frac{1}{6} = \frac{1}{6}$.

(b) Calculate the variance $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \left(1 \times \frac{2}{6} + 1 \times \frac{3}{6} + 4 \times \frac{1}{6}\right) - \left(\frac{1}{6}\right)^2 = \frac{2}{6} + \frac{1}{6} - \frac{1}{36} = \frac{35}{36} \approx 1.47$.

6. (a) If $0 < r < 1$, what is the sum of the infinite geometric series $1 + r + r^2 + r^3 + r^4 + \ldots$?

Answer $\frac{1}{1-r}$. The finite sum $1 + r + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$ which tends to $\frac{1}{1-r}$ as $n \to \infty$. 
(b) A p.m.f. of a random variable is given by the formula \( f(x) = c/3^x \) for \( x = 0, 1, 2, \ldots \). What is the value of \( c \)?

We need \( \sum f(x) = 1 \), so
\[
1 = c/3^0 + c/3^1 + \cdots = c(1 + (1/3) + (1/3)^2 + \cdots) = c/(1 - 1/3).
\]
Thus \( c = 1 - \frac{1}{3} = \frac{2}{3} \).

7. An urn contains ten balls. Six have the number 6 written on them, and the other four have the number 4 written on them. If you take 3 balls from the urn (without replacement), what is the expected sum of the numbers on the three balls?

Imagine taking all the balls out and putting them in a row. Then the \( i \)th ball has probability \( \frac{6}{10} \) of being a ‘6’ and \( \frac{4}{10} \) of being a ‘4’ as it is a uniformly random choice of ball. The mean is \( 6 \times \frac{6}{10} + 4 \times \frac{4}{10} = 5.2 \). Thus the sum of the first three has mean \( 3 \times 5.2 = 15.6 \). (Even though the balls are not independent.)

8. A store sells 3 brands of DVD players. The brands \( B_1, B_2, B_3 \) account for 50\%, 30\%, and 20\% of sales respectively of DVD players. The probabilities of a player needing repair within the warranty period are 8\%, 6\%, and 1\% for the three brands respectively. If a player is brought back for repair under warranty, what is the probability it is of brand \( B_1 \)?

Let \( R \) be the event that it is brought back for repair. Then
\[
\mathbb{P}(B_1 \mid R) = \frac{\mathbb{P}(R \mid B_1) \mathbb{P}(B_1)}{\sum \mathbb{P}(R \mid B_i) \mathbb{P}(B_i)} = \frac{.08 \times .50}{.08 \times .50 + .06 \times .30 + .01 \times .20} = \frac{.040}{.060} = \frac{2}{3}.
\]

9. Sears orders 1000 shirts to be shipped. 10 of them have defects. A certain store receives 50 of the shirts from this shipment.

(a) Using factorials or binomial coefficients, write down an expression for the probability that exactly 4 of the shirts received have defects.

\[
\frac{\binom{10}{4} \binom{990}{46}}{\binom{1000}{50}}
\]

(b) What is the name given to the distribution of the number of defective shirts received by the store?
Hypergeometric distribution

10. (a) Use the Binomial theorem to evaluate \( \sum_{r=0}^{n} \binom{n}{r} (\frac{1}{2})^r (\frac{3}{2})^{n-r} = (\frac{1}{2} + \frac{3}{2})^n = 2^n \).

(b) Let \( X \) be the number of heads obtained when tossing \( n \) fair coins. Evaluate \( \mathbb{E}(3^X) \).

\[
\mathbb{E}(3^X) = \sum_{r=0}^{n} 3^r \mathbb{P}(X = r) = \sum_{r=0}^{n} 3^r \binom{n}{r} (1/2)^r (1/2)^{n-r}
\]
\[
= \sum_{r=0}^{n} \binom{n}{r} (3/2)^r (1/2)^n = (3/2 + 1/2)^n = 2^n.
\]

Grading Scale:  