1. Let the random variable $X$ have p.d.f. $f_X(x) = \begin{cases} \frac{2}{x^3}, & \text{for } x \geq 1; \\ 0, & \text{otherwise}. \end{cases}$

(a) Find the cumulative distribution function of $X$.

\[
F(x) = \int_1^x \frac{2}{t^3} \, dt = -\frac{1}{x^2} \bigg|_1^3 = 1 - \frac{1}{x^2} \quad \text{for } x > 1. \quad \text{For } x \leq 1, \quad F(x) = 0.
\]

(b) Find the mean $\mathbb{E}(X)$.

\[
\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot \frac{2}{x^3} \, dx = \int_1^{\infty} \frac{2}{x^2} \, dx = -\frac{2}{x} \bigg|_{1}^{\infty} = 0 - (-2) = 2.
\]

(c) Show that $\text{Var}(X) = \infty$.

\[
\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \frac{2}{x^3} \, dx = \int_1^{\infty} \frac{2}{x^2} \, dx = 2 \log x \bigg|_1^{\infty} = \infty \quad \text{as } \log x \to \infty \text{ as } x \to \infty.
\]

Hence $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \mathbb{E}(X^2) - 4 = \infty$.

2. The moment generating function of $X$ is given by $M_X(t) = e^{3(1+t)}$.

(a) Find $\mathbb{E}(X)$.

\[
M_X(t) = e^{3t^2}, \quad M_X'(t) = (3 + 6t)e^{3t^2}, \quad \text{so } \mathbb{E}(X) = M_X'(0) = 3e^0 = 3.
\]

(b) Find $\text{Var}(X)$.

\[
M_X''(t) = 6e^{3t^2} + (3 + 6t)(3 + 6t)e^{3t^2}, \quad \text{so } \mathbb{E}(X^2) = M_X''(0) = 6 + 9 = 15.
\]

Hence $\text{Var}(X) = 15 - 3^2 = 6$.

Alternatively, $\text{Var}(X) = \frac{d^2}{dt^2} \log M_X(t)\bigg|_{t=0} = \frac{d^2}{dt^2}(3t + 3t^2)\bigg|_{t=0} = 6.$

3. Sick leave time of employees at a firm for a month is normally distributed with mean 200 hours and variance 400 hours².

(a) Find the probability that total sick leave for one month will be less than 150 hours.

Let $X$ be the total sick leave for one month so that $X$ has a $N(200, 400)$ distribution. Then $Z = (X - 200)/\sqrt{400} = (x - 200)/20$ has a $N(0, 1)$ distribution. We need $\mathbb{P}(X < 150) = \mathbb{P}(Z < -50/20) = \mathbb{P}(Z < -2.5) = \Phi(-2.5).$ Looking this up in the tables gives .00621.

(b) In planning schedules, how much time should be budgeted for sick leave if that amount will only be exceeded with probability 0.10?

We need $\mathbb{P}(X > x) = 0.10$ or $\mathbb{P}(X < x) = 0.90.$ Looking in the tables gives $\mathbb{P}(Z < 1.28) = \Phi(1.28) = 0.89973$ as the closest value. Now $X = 200 + 20Z$, so the limit is about $200 + 20(1.28) = 225.6$ hours.

4. Suppose $X$ has a uniform $U(-1, 1)$ distribution (p.d.f. $\frac{1}{2}$ for $-1 \leq x \leq 1$).

(a) Find $\mathbb{E}(X), \mathbb{E}(X^2), \mathbb{E}(X^3), \mathbb{E}(X^4)$.

\[
\mathbb{E}(X^n) = \int_{-1}^{1} x^n \frac{1}{2} \, dx = \frac{1}{2(n+1)} |x^{n+1}|_{-1}^{1} = \frac{1}{2(n+1)} (1 - (-1)^{n+1}).
\]

Hence $\mathbb{E}(X) = 0, \mathbb{E}(X^2) = \frac{1}{3}, \mathbb{E}(X^3) = 0, \mathbb{E}(X^4) = \frac{1}{5}.$
(b) Find the correlation coefficient $\rho$ between $X$ and $Y$ where $Y = X^2$.
\[ \text{Cov}(X,Y) = E(XY) - E(X)E(Y) = E(X^3) - E(X)E(X^2) = 0 - 0 = 0. \]
Hence $\rho = \text{Cov}(X,Y)/\sqrt{\cdots} = 0$.

(c) Are $X$ and $Y$ independent? Explain.
No. For example, if we condition on $X = .5$ then $P(Y = .25) = 1$, whereas this is not true if we don’t condition. Alternatively, $P(|X| < .1) > 0$ and $P(|Y| > .9) > 0$, but $P(|X| < .1$ and $|Y| > .9) = 0$, which is not the product of $P(|X| < .1)$ and $P(|Y| > .9)$.

5. Suppose $X$ has a cumulative distribution $F(x) = 2x - x^2$ for $0 \leq x \leq 1$.

(a) What is the p.d.f. of $X$?
\[ f(x) = F'(x) = 2 - 2x \quad \text{for } 0 < x < 1 \quad \text{and } f(x) = 0 \quad \text{otherwise}. \]

(b) Suppose $Y = \sqrt{X}$. What is the cumulative distribution of $Y$?
\[ F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y) = P(X \leq y^2) = F(y^2) = 2y^2 - y^4 \quad \text{for} \quad 0 \leq y^2 \leq 1 \quad \text{and } y \geq 0. \]
Hence $F_Y(y) = 2y^2 - y^4$ for $0 \leq y \leq 1$, $F_Y(y) = 0$ for $y < 0$, and $F_Y(y) = 1$ for $y > 1$.

6. The conditional p.d.f. of $X$ given $Y$ is $f_{X|Y}(x|y) = \begin{cases} ae^{-xy}, & \text{for } x > 0; \\ 0, & \text{otherwise.} \end{cases}$

The marginal distribution of $Y$ is $f_Y(y) = \begin{cases} 5e^{-5y}, & \text{for } y > 0; \\ 0, & \text{otherwise.} \end{cases}$

(a) Find $a$.
\[ 1 = \int_{-\infty}^{\infty} f_{X|Y}(x|y) dx = \int_{0}^{\infty} ae^{-xy} dx = -\frac{a}{y} e^{-xy}|_{0}^{\infty} = \frac{a}{y} \quad \text{when } y > 0. \]
Hence $a = y$.

(b) Find the joint p.d.f. of $X$ and $Y$.
\[ f_{X,Y}(x,y) = f_Y(y) f_{X|Y}(x|y) = (5e^{-5y})(ye^{-xy}) = 5ye^{-(5+x)y} \quad \text{for } x, y > 0; \quad \text{and } 0 \quad \text{otherwise.} \]

(c) Find the marginal distribution $f_X(x)$ of $X$.
\[ f_X(x) = \int_{0}^{\infty} f_{X,Y}(x,y) dy = \int_{0}^{\infty} 5ye^{-(5+x)y} dy. \]
We integrate by parts:
\[ f_X(x) = (5y)(\frac{-1}{5+x}e^{-(5+x)y})|_{0}^{\infty} - \int_{0}^{\infty} 5(-\frac{1}{5+x}e^{-(5+x)y}) dy \]
\[ = (0-0) + \frac{5}{5+x} \int_{0}^{\infty} e^{-(5+x)y} dy = \frac{5}{5+x} (\frac{1}{1}e^{-(5+x)y})|_{0}^{\infty} = \frac{5}{5+x} \frac{1}{5+x} \quad \text{(note } 5+x > 0). \]
Hence $f_X(x) = \frac{5}{(5+x)^2}$ for $x > 0$ and $0$ otherwise.


[Will double all grades when calculating course totals.]