Math 7411 1. Topologies Spring 2008

Definition A topology on a set X, is a collection $\mathcal{T}$ of subsets of X (the *open* sets) satisfying:

T1. The sets $\emptyset$ and X are open: $\emptyset, X \in \mathcal{T}$.

T2. Any *finite* intersection of open sets is open: $U_1, \ldots, U_n \in \mathcal{T} \Rightarrow \bigcap_{i=1}^n U_i \in \mathcal{T}$.

T3. The union of *any* collection of open sets is open: $\mathcal{C} \subseteq \mathcal{T} \Rightarrow \bigcup_{U \in \mathcal{C}} U \in \mathcal{T}$.

Note that it is enough in T2 to assume just that the intersection of two open sets is open, the general case following by induction. However, it is important in T3 that any union, even infinite or uncountable, is allowed.

Examples

1. The *discrete* topology on X: every subset of X is open.
2. The *indiscrete* topology on X: only $\emptyset$ and X are open.
3. The *co-finite* topology on X: U is open iff $X \setminus U$ is a finite set.
4. The usual topology on $\mathbb{R}$. More generally, any *metric space*:

Definition A *metric space* is a set X and a "distance" function $d\colon X \times X \to \mathbb{R}$ such that

M1. $d(x, y) \geq 0$ with equality iff $x = y$,

M2. $d(x, y) = d(y, x)$,

M3. $d(x, z) \leq d(x, y) + d(y, z)$.

Define, for any metric space (X, d), a topology $\mathcal{T}$ on X by declaring a set $U \subseteq X$ open iff $\forall x \in U \colon \exists \varepsilon > 0 \colon \forall y \in X \colon d(x, y) < \varepsilon \Rightarrow y \in U$. [Check this is a topology.]

In general, it can be rather awkward to specify what is an arbitrary open set in a topology, so it is often convenient to define a small collection of open sets from which all others can be constructed.

Definition A *base* for a topology $\mathcal{T}$ is a collection $\mathcal{B}$ of open sets such that every open set U can be written as a union of elements of $\mathcal{B}$. Equivalently, $\forall x \in U \colon \exists B \in \mathcal{B} \colon x \in B \subseteq U$.

Lemma *Given any metric space* (X, d)*, the set of* open balls $B(x, r) = \{y \mid d(y, x) < r\}$, $x \in X$, $r > 0$, *forms a base for the topology induced by* (X, d).

Examples

1. The collection of singleton sets $\{\{x\} \mid x \in X\}$ forms a base for the discrete topology.
2. The single set X forms a base for the indiscrete topology.

3. Given any totally ordered set $(X, \leq)$ we can define the *order topology* by giving as a basis the collection of all open intervals $(a,b) = \{x \mid a < x < b\}$, $(-\infty, b) = \{x \mid x < b\}$, $(a, \infty) = \{x \mid x > a\}$. Note that the order topology given by the usual ordering on $\mathbb{R}$ is the same as the standard (metric space) topology on $\mathbb{R}$.

Lemma *A collection $\mathcal{B}$ of subsets of a set X is a base for some topology iff $\bigcup_{B\in\mathcal{B}} B = X$ and for every $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.*

Definition Given two topologies $\mathcal{T}$ and $\mathcal{T}'$ on the same set with $\mathcal{T} \subseteq \mathcal{T}'$, we say $\mathcal{T}'$ is *finer* than $\mathcal{T}$, or $\mathcal{T}$ is *coarser* than $\mathcal{T}'$.

Lemma *If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X, $\mathcal{B}$ is a base for $\mathcal{T}$, and $\mathcal{B}'$ is a base for $\mathcal{T}'$, then $\mathcal{T}'$ is finer than $\mathcal{T}$ iff $\forall x \in B \in \mathcal{B}\colon \exists B' \in \mathcal{T}'\colon x \in B' \subseteq B$.*

Examples

1. The discrete topology on X is finer than any other topology on X, the indiscrete topology on X is coarser than any other topology on X.
2. The *lower limit* topology on $\mathbb{R}$, defined by the base consisting of all half-intervals $[a,b)$, $a, b \in \mathbb{R}$, is finer than the usual topology on $\mathbb{R}$.
3. The topology on $\mathbb{R}^2$ defined by the base consisting of all rectangles $(a,b) \times (c,d)$ is the same as the topology given by the usual Euclidean metric. [Prove both are finer than the other.]

Definition A *subbase* for a topology $\mathcal{T}$ is a collection $\mathcal{S}$ of open sets such that $\mathcal{T}$ is the coarsest topology such that $\mathcal{S} \subseteq \mathcal{T}$. Equivalently, $\mathcal{T}$ consists of X and all unions of finite intersections of elements of $\mathcal{S}$.

Unlike for bases, any collection of subsets of X can act as a subbase for some topology.

Examples The collection of *rays* $(-\infty, a)$ and (a, ∞) form a subspace for the standard topology on $\mathbb{R}$ (or any other order topology). The collection of horizontal and vertical strips $\mathbb{R} \times (a,b)$ and $(a,b) \times \mathbb{R}$ form a subbase for the standard topology on $\mathbb{R}^2$.

A subset $C \subseteq X$ is *closed* iff $X \setminus C$ is open. Note that $\emptyset$, X, finite unions of closed sets, and arbitrary intersections of closed sets are all closed.

The *interior* of a set S, $S^\circ = \bigcup_{U \text{ open}, U\subseteq S} U$, is the largest open set contained in S.
The *closure* of a set S, $\bar{S} = \bigcap_{C \text{ closed}, C\supseteq S} C$, is the smallest closed set containing S.
The *boundary* ∂S of a set S is $\partial S = \bar{S} \setminus S^\circ$.
Note that S is open iff $S^\circ = S$ and S is closed iff $\bar{S} = S$.

If $\mathcal{B}$ is a base for the topology then $\bar{S} = \{x \mid \forall B \in \mathcal{B}\colon x \in B \Rightarrow B \cap S \neq \emptyset\}$, $S^\circ = \{x \mid \exists B \in \mathcal{B}\colon x \in B \subseteq S\}$, and $\partial S = \{x \mid \forall B \in \mathcal{B}\colon x \in B \Rightarrow B \cap S \neq \emptyset \text{ and } B \cap (X \setminus S) \neq \emptyset\}$.

Math 7411 2. Limits and continuity Spring 2008

Definition An *(open) neighborhood* of a point x is an open set containing x.

Note: Some people define a *neighborhood* of a point to be any set containing an open set containing x, but we shall define it as being the same as open neighborhood. I will write "open $U \ni x$" for "U is an open neighborhood of x".

Definition A *base of neighborhoods* of a point x is a set $\mathcal{B}_x$ of open neighborhoods of x such that for any open $U \ni x$, there is a $B \in \mathcal{B}_x$ such that $B \subseteq U$.

Example The set $\{B(x,r) \mid r > 0\}$ of open balls about x in a metric space.

Definition A function $f\colon X \to Y$ between two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ is *continuous* if for every open set V of Y, $f^{-1}[V]$ is open in X.
A function $f\colon X \to Y$ between $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ is *continuous at x* if for every open $V \ni f(x)$, there exists an open $U \ni x$ with $f[U] \subseteq V$.

Note that it is enough to check the conditions with V an element of some base (or subbase) of Y, or some base of neighborhoods of $f(x)$.

Lemma *For functions $f\colon X \to Y$ between metric spaces, f is continuous at x iff* $\forall \varepsilon > 0\colon \exists \delta > 0\colon \forall y \in X\colon d_X(y,x) < \delta \Rightarrow d_Y(f(y), f(x)) < \varepsilon$.

Proof. $\Leftarrow$: The condition is equivalent to $f[B_X(x,\delta)] \subseteq B_Y(f(x),\varepsilon)$, which is enough since every open $V \ni f(x)$ contains some $B_Y(f(x),\varepsilon)$. $\Rightarrow$: Take $V = B_Y(f(x),\varepsilon)$, and $f[U] \subseteq V$. Then openness of U implies that $\exists \delta > 0\colon B_X(x,\delta) \subseteq U$, so $f[B_X(x,\delta)] \subseteq B_Y(f(x),\varepsilon)$. $\square$

Lemma *The following are equivalent for a function $f\colon X \to Y$ between topological spaces.*

1. *f is continuous.*
2. *$f^{-1}[C]$ is closed for every closed set C.*
3. *f is continuous at x for all $x \in X$.*
4. *$f[\bar{A}] \subseteq \overline{f[A]}$ for every $A \subseteq X$.*

Proof. 1$\Leftrightarrow$2: $f^{-1}[Y \setminus S] = X \setminus f^{-1}[S]$. 2$\Rightarrow$4: $f^{-1}[\overline{f[A]}]$ is closed and contains A, so contains $\bar{A}$. Thus $f[\bar{A}] \subseteq \overline{f[A]}$. 4$\Rightarrow$2: Let $A = f^{-1}[C]$ with C closed, then $f[\bar{A}] \subseteq \overline{f[A]} \subseteq \bar{C} = C$ thus $\bar{A} \subseteq f^{-1}[C] = A$ and A is closed. 1$\Rightarrow$3: If $V \ni f(x)$ is open then $f^{-1}[V]$ is open and contains x. 3$\Rightarrow$1: $f^{-1}[V] = \bigcup_{x \in f^{-1}[V]} \bigcup_{\text{open } U \ni x,\, f[U] \subseteq V} U$, which is open. $\square$

Lemma *Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are functions between topological spaces.*

1. *If f and g are continuous, then so is $g \circ f\colon X \to Z$.*
2. *If f is continuous at x and g is continuous at $f(x)$ then $g \circ f$ is continuous at x.*

Proof. For part 2, if $W \ni g(f(x))$ is open, continuity of g at $f(x)$ implies that there is an open $V \ni f(x)$ with $g[V] \subseteq W$, and continuity of f at x implies that there is an open $U \ni x$ with $f[U] \subseteq V$. But then $g \circ f[U] \subseteq g[V] \subseteq W$, so $g \circ f$ is continuous x. Part 1 follows by letting x vary over X, or directly by noting that $(g \circ f)^{-1}[W] = f^{-1}[g^{-1}[W]]$ is open. □

Definition A function $f\colon X \to Y$ between two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ is a *homeomorphism* (and X and Y are homeomorphic) if it is continuous and there exists a continuous inverse function $g\colon Y \to X$, $g \circ f = 1_X$, $f \circ g = 1_Y$.

Note that it is *not* enough for f to be a bijective continuous map. Homeomorphisms are the "isomorphisms" of topological spaces, in that if two spaces are homeomorphic, then they are the "same" space up to renaming of the elements of the set.

Examples

1. The identity map $(X, \mathcal{T}) \to (X, \mathcal{T}')$ is continuous iff $\mathcal{T}$ is finer than $\mathcal{T}'$ and a homeomorphism iff $\mathcal{T} = \mathcal{T}'$.
2. The function $f\colon [0, 2\pi) \to \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ given by $f(t) = (\cos t, \sin t)$ is bijective and continuous but is not a homeomorphism. (Usual metric on both spaces.)

Limits of sets and sequences

Definition A *limit point* of a set A is a point $x \in X$ such that every open $U \ni x$ intersects A in a point other than x. Equivalently x is a limit point of A iff $x \in \overline{A \setminus \{x\}}$.

Every point of $\bar{A}$ is either a limit point of A, or an *isolated point* of A, i.e., a point $x \in A$ such that some open neighborhood of x intersects A only in x.

Definition A *limit* of the sequence $(x_n)_{n\in\mathbb{N}}$ is a point x such that every open $U \ni x$ contains all but a finite number of terms x_n, i.e., $\exists n_0\colon \forall n \geq n_0\colon x_n \in U$. In this case we write $x_n \to x$.

Examples

1. If $x_n = c$ is the constant sequence in $\mathbb{R}$ (with the standard topology), then c is a limit of the sequence $(x_n)_{n\in\mathbb{N}}$, but is not a limit point of the set $\{x_n \mid n \in \mathbb{N}\}$.
2. If $x_n = \frac{1}{n}$ (n odd), $x_n = 1$ (n even), then there is no limit (in $\mathbb{R}$ with the standard topology) of the sequence $(x_n)_{n\in\mathbb{N}}$, but the set $\{x_n \mid n \in \mathbb{N}\}$ has the unique limit point 0.
3. In the indiscrete topology, every point is a limit of every sequence.
4. If we give $\mathbb{N} \cup \{\omega\}$ the order topology, then $x_n \to x$ iff the function $f\colon \mathbb{N} \cup \{\omega\} \to X$ given by $f(n) = x_n$, $f(\omega) = x$, is continuous.

Math 7411 3. New spaces from old Spring 2008

Subspaces

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then the *subspace* (or *relative* or *induced*) topology $\mathcal{T}_{\subseteq} = \{U \cap A \mid U \in \mathcal{T}\}$ is a topology on A.

Note that open sets $U \cap A \in \mathcal{T}_{\subseteq}$ may not be open in X. We say $U \cap A$ is open *relative* to A, or open *in* A. The closed sets in A are sets of the form $C \cap A$ where C is closed in X.

1. $\mathcal{T}_{\subseteq}$ is the coarsest topology such that the inclusion map $i\colon A \to X$ is continuous.
2. $f\colon Y \to X$ can be written (uniquely) as $f = i \circ \tilde{f}$ iff $f[Y] \subseteq A$. In this case $f\colon Y \to X$ is continuous iff $\tilde{f}\colon Y \to A$ is continuous.

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ & {\scriptstyle \tilde{f}}\searrow & \uparrow i \\ & & A \end{array}$$

3. If $f\colon X \to Y$ is continuous (at $x \in A$), then $f_{|A}\colon A \to Y$ is continuous (at x).
4. If A is **open** and $f_{|A}\colon A \to Y$ is continuous at x then $f\colon X \to Y$ is continuous at x.
5. If $S \subseteq A$ then the closure of S in A is $\bar{S} \cap A$. [Interior $= (S \cup (X \setminus A))^\circ \cap A$.]
6. A subspace topology of a subspace topology is a subspace topology of the original.
7. The subspace topology on a subset of a metric space is the same as the metric topology obtained by restricting the metric to the subspace.

Note that a subspace of an order topology may not be the order topology on the subset, e.g., in $[0, 1) \cup \{2\} \subseteq \mathbb{R}$, $\{2\}$ is open in the subset topology but not in the order topology.

Pasting lemma *Suppose $f\colon X \to Y$ is a function between topological spaces.*

1. *If $\{U_\alpha \mid \alpha \in I\}$ is an arbitrary collection of open subsets of X that cover X: $\bigcup U_\alpha = X$, and each $f_{|U_\alpha}\colon U_\alpha \to Y$ is continuous, then f is continuous.*
2. *If $\{C_1, \dots, C_n\}$ is a* **finite** *collection of closed subsets of X that cover X: $\bigcup_{i=1}^n C_i = X$, and each $f_{|C_i}\colon C_i \to Y$ is continuous, then f is continuous.*

Quotients

If $\sim$ is an equivalence relation on X then define the *quotient topology* $\mathcal{T}_Q$ on $X/{\sim}$, by U open in $X/{\sim}$ iff $\pi^{-1}[U]$ is open in X. Here $\pi\colon X \to X/{\sim}$ is defined by $\pi(x) = \bar{x}$.

1. $\mathcal{T}_Q$ is the finest topology on $X/{\sim}$ such that $\pi\colon X \to X/{\sim}$ is continuous.
2. $f\colon X \to Y$ can be written (uniquely) as $f = \tilde{f} \circ \pi$ iff $x \sim x' \Rightarrow f(x) = f(x')$. In this case $f\colon X \to Y$ is continuous iff $\tilde{f}\colon X/{\sim} \to Y$ is continuous.

$$\begin{array}{ccc} Y & \xleftarrow{f} & X \\ & {\scriptstyle \tilde{f}}\nwarrow & \downarrow \pi \\ & & X/{\sim} \end{array}$$

If $p\colon X \to Q$ is surjective, then we call p a *quotient map* if $p^{-1}[U]$ open $\Leftrightarrow$ U open. In this case Q is homeomorphic to $X/{\sim}$ where $x \sim x'$ iff $p(x) = p(x')$.

As a special case, if $A \subseteq X$, write $X/A = X/{\sim}$ where $x \sim y$ if $x, y \in A$ or $x = y$. The space X/A "shrinks" A down to a single point.

Products

If $(X_\alpha, \mathcal{T}_\alpha)$, $\alpha \in I$, are spaces, the *product topology* $\mathcal{T}_\Pi$ on $X = \prod X_\alpha$ is given by the base of *cylinder sets*: $\{\prod U_\alpha \mid U_\alpha \text{ open in } X_\alpha,\ U_\alpha = X_\alpha \text{ for all but a finite number of } \alpha\}$, or equivalently by the subbase of *elementary cylinder sets*: $U_\alpha \times \prod_{\beta \neq \alpha} X_\beta$, U_α open in X_α.

One can also define the *box topology* $\mathcal{T}_\Box$ on $X = \prod X_\alpha$ by giving a base of product sets $\prod U_\alpha$, U_α open in X_α. This topology however is not as nice as $\mathcal{T}_\Pi$ when there are an infinite number of factors, so if no topology is specified on $\prod X_\alpha$, then the product topology should be assumed. Box and product topologies are the same for finite products. In both box and product topologies, one may restrict U_α to lie in a base of the topology on X_α.

1. $\mathcal{T}_\Pi$ is the coarsest topology such that each projection map $\pi_\alpha \colon X \to X_\alpha$ is continuous.
2. If $f_\alpha \colon Y \to X_\alpha$ are given, then there is a unique $f \colon Y \to \prod X_\alpha$ with $f_\alpha = \pi_\alpha \circ f$, namely $f(y) = (f_\alpha(y))_{\alpha \in I}$. Moreover f is continuous iff each f_α is continuous.
$$\begin{array}{ccc} Y & \xrightarrow{f_\alpha} & X_\alpha \\ & {}_f\searrow & \uparrow \pi_\alpha \\ & & \prod X_i \end{array}$$
3. (Special case of 2.) Considering an element of $\prod_{\alpha \in I} X$ as a function $g \colon I \to X$, $g_n \to g$ in the product topology iff $g_n(\alpha) \to g(\alpha)$ for all $\alpha \in I$.
4. If $A_\alpha \subseteq X_\alpha$ then the product (or box) topology of the subspace topologies on $\prod A_\alpha$ is the same as the subspace topology of the product (or box) topology on $\prod X_\alpha$.
5. If $A_\alpha \subseteq X_\alpha$ then $\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$ in both product and box topologies.
6. The projection π_α is a quotient map from $\prod X_\alpha$ to X_α.

It is not so simple to determine whether a map $f \colon \prod X_\alpha \to Y$ is continuous. In particular, f can be continuous in each coordinate separately without being continuous.

Disjoint Unions (Direct sums)

If $(X_\alpha, \mathcal{T}_\alpha)$ are (disjoint) topological spaces, the *disjoint union* or *direct sum* topology on $X = \bigcup X_\alpha$ is given by $\mathcal{T}_\cup = \{S \subseteq X \mid \text{ every } S \cap X_\alpha \text{ is open } \} = \{\bigcup U_\alpha \mid U_\alpha \text{ open in } X_\alpha\}$.

1. $\mathcal{T}_\cup$ is the finest topology such that each inclusion $i_\alpha \colon X_\alpha \to X$ is continuous.
2. If $f_\alpha \colon X_\alpha \to Y$ are given, then there is a unique $f \colon \bigcup X_\alpha \to Y$ with $f_\alpha = f \circ i_\alpha$, namely $f(y) = f_\alpha(y)$ for $y \in X_\alpha$. Moreover f is continuous iff each f_α is continuous.
$$\begin{array}{ccc} Y & \xleftarrow{f_\alpha} & X_\alpha \\ & {}_f\nwarrow & \downarrow i_\alpha \\ & & \bigcup X_\alpha \end{array}$$
3. Each X_α is both open and closed in $\bigcup X_\alpha$.
4. The subspace topology on $X_\alpha \subseteq X$ is the original topology on X_α.

Math 7411 4. Connectivity Spring 2008

We call a pair (U, V) of disjoint non-empty open sets with $U \cup V = X$ a *separation* of X. The space X is *connected* if there is no separation of X. A subset of X is connected if it is connected in the subspace topology.

Lemma 4.1 *The following are equivalent for a topological space $(X, \mathcal{T})$.*

1. *X is connected.*
2. *There is no proper non-empty set S, $\emptyset \subsetneq S \subsetneq X$, that is both open and closed.*
3. *Every continuous map from X to the discrete space $\{0, 1\}$ is constant.*
4. *X is not the topological disjoint union (direct sum) of two non-empty spaces.*

Proof. 1⇒2: if S is non-empty proper open and closed, then $(S, X \setminus S)$ separates X. 2⇒3: if $f\colon X \to \{0, 1\}$ is continuous, non-constant, then $S = f^{-1}[\{0\}]$ is open, closed, $\neq \emptyset, X$. 3⇒4: If $X = X_1 \cup X_2$ then defining $f(x) = 1$ on X_1 and $f(x) = 0$ on X_2 gives a non-constant continuous map to $\{0, 1\}$ (continuous by universal property of disjoint unions). 4⇒1: If (U, V) separates X then $A \subseteq X$ is open $\Rightarrow A \cap U$ and $A \cap V$ are open (since U and V are open), while if $A \cap U$ and $A \cap V$ are open then $A = (A \cap U) \cup (A \cap V)$ is open. Thus X is the topological disjoint union of the subspaces U and V. □

A *path* from x to y in a space X is a continuous map $\gamma\colon [0, 1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$. A space $(X, \mathcal{T})$ is *path connected* if $\forall x, y \in X\colon \exists$ a path from x to y in X.

Lemma 4.2 *Any continuous image of a (path) connected space is (path) connected.*

Proof. Assume $f\colon X \to Y$ is continuous, so $f\colon X \to f[X]$ is also continuous
1. If $g\colon f[Y] \to \{0, 1\}$ is continuous then so is $gf\colon X \to \{0, 1\}$. But then gf is constant (as X is connected), so g is constant ($f\colon X \to f[X]$ surjective) and so Y is connected.
2. Assume $x, y \in f[X]$, say $x = f(x')$ and $y = f(y')$. Since X is path connected, $\exists$ continuous $\gamma\colon [0, 1] \to X$ with $\gamma(0) = x'$, $\gamma(1) = y'$. Then $\gamma f\colon [0, 1] \to f[Y]$ is continuous with $\gamma f(0) = x$, $\gamma f(1) = y$. □

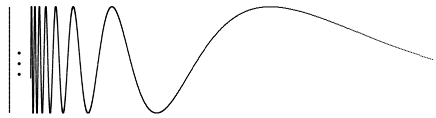
Lemma 4.3 *If A_α are (path) connected and $\bigcap A_\alpha \neq \emptyset$ then $\bigcup A_\alpha$ is (path) connected.*

Proof. 1. Fix $p \in \bigcap A_\alpha$ and suppose $g\colon \bigcup A_\alpha \to \{0, 1\}$ is continuous. If $x \in \bigcup A_\alpha$ then $x \in A_\alpha$ for some α. But then $p \in A_\alpha$, $g_{|X_\alpha}\colon X_\alpha \to \{0, 1\}$ is continuous, so constant, and so $g(x) = g(p)$. Since this holds for all x, g is constant, and so $\bigcup A_\alpha$ is connected.
2. Fix $p \in \bigcap A_\alpha$. If $x \in \bigcup A_\alpha$ then $x \in A_\alpha$ for some α. But then $p \in A_\alpha$ so there exists a path γ from p to x in A_α, and hence in $\bigcup A_\alpha$. Similarly, if $y \in \bigcup A_\alpha$, then $\exists$ path γ' from p to y. Then $\gamma''(t) = \gamma(1 - 2t)$, $t < \frac{1}{2}$, and $\gamma'(2t - 1)$, $t \geq \frac{1}{2}$, is a path from x to y. □

We shall see below that intervals in $\mathbb{R}$ with the standard topology are connected. Thus

Lemma 4.4 *Path connectivity ⇒ connectivity.*

Proof. If X is path connected, then it is the union of the images of the paths from p to x, as x ranges over X and $p \in X$ is fixed. Each of these subsets is the continuous image of the connected set $[0, 1]$ and intersect at p. $\square$



The converse is not true: Define the *Topologist's sine curve* as $X = X_1 \cup X_2$ where $X_1 = \{0\} \times [-1, 1]$, $X_2 = \{(x, \sin(1/x)) \mid x > 0\}$, all as subspaces of $\mathbb{R}^2$.

The subspaces X_1 and X_2 are both connected (continuous images of $[-1, 1]$ and $(0, \infty)$). Thus if $g\colon X \to \{0, 1\}$ is continuous then g is constant on X_1 and on X_2. But there exists a sequence of points $x_n = (\frac{1}{\pi n}, 0) \in X_2$ converging to $x = (0, 0) \in X_1$ and $g^{-1}[\{g(x)\}]$ is open and contains x, so contains some (indeed most) x_n. Hence $g(x_n) = g(x)$ and g is constant. However X is not path connected: Suppose there is a path γ from some point in X_1 to some point in X_2. Let $t = \sup\{t \mid \gamma(t) \in X_1\}$. Since X_1 is closed, $\gamma^{-1}[X_1]$ is closed, and so $\gamma(t) \in X_1$. Since γ is continuous, there exists an $\varepsilon > 0$, $t + \varepsilon < 1$, such that $\gamma[[t, t + \varepsilon]] \subseteq B(\gamma(t), 0.1) \cap X$. But $\gamma(t + \varepsilon) \in B(\gamma(t), 0.1) \cap X_2$, so $\gamma(t + \varepsilon)$ lies in a segment of the sine curve in $B(\gamma(t), 0.1) \cap X_2$ that is clearly both open and closed in the subspace topology. But $\gamma_{|[t,t+\varepsilon]}$ is a path to this point from X_1.

Lemma 4.5 *The product of (path) connected spaces is (path) connected.*

Proof. 1. Suppose $g\colon \prod X_\alpha \to \{0, 1\}$ is continuous. Pick $x = (x_\alpha)$, $y = (y_\alpha)$ in $\prod X_\alpha$. Let $U = \prod U_\alpha$ be a basic neighborhood of y inside the open set $g^{-1}[\{g(y)\}]$. Let $I = \{a_1, \dots, a_n\}$ be the (finite) set of indices where $U_\alpha \neq X_\alpha$. Define $x^{(k)}$ by $(x^{(k)})_\alpha = y_\alpha$ if $\alpha \in \{\alpha_1, \dots, \alpha_k\}$ and $(x^{(k)})_\alpha = x_\alpha$ otherwise (so $x^{(k)}$ differs from $x^{(k-1)}$ in just the α_k coordinate). The map $i_k\colon X_{\alpha_k} \to \prod X_\alpha$ given by $(i_k(x))_\alpha = (x^{(k)})_\alpha$ when $\alpha \neq \alpha_k$ and $(i_k(x))_{\alpha_k} = x$ is continuous (check inverse image of any basic open set is open). Now $i_k(x_{\alpha_k}) = x^{(k-1)}$ and $i_k(y_{\alpha_k}) = x^{(k)}$. Since X_{α_k} is connected and $gi_k\colon X_{\alpha_k} \to \{0, 1\}$ is continuous, $g(i_k(x_{\alpha_k})) = g(i_k(y_{\alpha_k}))$ and so $g(x^{(k-1)}) = g(x^{(k)})$. But $x^{(0)} = x$ and $x^{(n)} \in U \subseteq g^{-1}[\{g(y)\}]$, so $g(x) = g(x^{(0)}) = \dots = g(x^{(n)}) = g(y)$.
2. If $x = (x_\alpha)$ and $y = (y_\alpha)$ then pick paths γ_α from x_α to y_α in X_α. Now $\gamma = (\gamma_\alpha)$ is continuous, so is a path from x to y in $\prod X_\alpha$. $\square$

Note: The box topology on a product of connected sets is not in general connected. For example, if U is the set of bounded sequences $(a_n) \in \prod_{n\in\mathbb{N}} \mathbb{R}$ and V is the set of unbounded sequences, then (U, V) separates $\prod_{n\in\mathbb{N}} \mathbb{R}$ in the box topology. Thus although $\mathbb{R}$ is connected and path connected, the box topology $\prod_{n\in\mathbb{N}} \mathbb{R}$ is not connected (and hence not path connected).

Lemma 4.6 *If $A \subseteq X$ is connected then so is $\bar{A}$.*

Proof. If $g\colon \bar{A} \to \{0, 1\}$ is continuous then $g_{|A}\colon A \to \{0, 1\}$ is continuous, thus constant and so $g[\bar{A}] \subseteq \overline{g[A]}$ is a single point. $\square$

Note: This is not true for path connectedness: the set X_2 in the Topologist's sine curve example is path connected but $\bar{X}_2 = X$ is not.

A totally ordered set $(X, \leq)$ satisfies the *least upper bound axiom* (LUB) if every nonempty subset $S \subseteq X$ that has some upper bound in X has a least upper bound in X. We write this least upper bound as $\sup S$. A *continuum* is a totally ordered set X, $|X| > 1$, satisfying the LUB axiom and such that $\forall x, y \colon x < y \Rightarrow \exists z \colon x < z < y$.

Note that $(X, \leq)$ satisfies LUB iff it satisfies the *greatest lower bound axiom*: every nonempty subset $S \subseteq X$ that has some lower bound in X has a greatest lower bound in X. [Proof: consider sup{lower bounds}.] We write this greatest lower bound as $\inf S$.

A subset S of a totally ordered set X is an *interval* iff $\forall a, b, c,\ a < b < c \colon a, c \in S \Rightarrow b \in S$. If X satisfies the LUB axiom then every interval can be written in one of the forms:
(a, b), $[a, b)$, $(a, b]$, $[a, b]$, $(-\infty, a)$, $(-\infty, a]$, (a, ∞), $[a, \infty)$, X, or $\emptyset$.
In general however, this is not true, e.g., $(\sqrt{2}, \pi) \cap \mathbb{Q}$ in $\mathbb{Q}$.

Examples

1. The real numbers $\mathbb{R}$ form a continuum.
2. Any well ordered set satisfies the LUB axiom, but is not a continuum. So does $\mathbb{Z}$.
3. Any interval in an ordered set satisfying the LUB axiom also satisfies the LUB axiom. Any interval in a continuum that contains at least 2 points is a continuum.

Lemma 4.7 *Suppose $(X, \leq)$ satisfies the LUB axiom, $X \neq \emptyset$, and let $\tilde{X} = \{(x, t) \mid x \in X,\ t \in [0, 1]$, and $t < 1$ if there is a least element $> x\} \subseteq X \times [0, 1]$. If $\tilde{X}$ is given the lexicographic ordering then $\tilde{X}$ is a continuum.*

Proof. If $S \subseteq \tilde{X}$ is nonempty and has an upper bound, say (x, t), then the projection $\{x \mid (x, t) \in S\}$ onto the 1st coordinate is nonempty and has the upper bound x. Let $x_0 = \sup\{x \mid (x, t) \in S\}$. If there is no t with $(x_0, t) \in S$ then $(x_0, 0)$ is a least upper bound. Otherwise let $t_0 = \sup\{t \mid (x_0, t) \in S\}$. If there is a least element $x_1 > x_0$ and $t_0 = 1$ then $(x_1, 0)$ is a least upper bound, otherwise (x_0, t_0) is a least upper bound. The other conditions for a continuum are easy to check. □

Examples

1. $\mathbb{Z} \times [0, 1)$ is order isomorphic to $\mathbb{R}$ and is a continuum.
2. The *ordered square* $[0, 1] \times [0, 1]$ is a continuum.
3. The *long line* $\Omega \times [0, 1)$, where Ω is the first uncountable ordinal, is a continuum.

Lemma 4.8 *Suppose $(X, \leq)$ is totally ordered and is given the order topology. If the subspace $S \subseteq X$ (with the subspace topology) is connected then S is an interval. If X is a continuum and S is an interval then S is connected.*

Proof. If S is not an interval, then there exist $a < b < c$ with $a, c \in S$, $b \notin S$. But then $U = (-\infty, b) \cap S$ and $V = (b, \infty) \cap S$ separate S.

Suppose now that X is a continuum, S is an interval, and (U, V) is a separation of S. Pick $a \in U$, $c \in V$ and assume wlog that $a < c$. Let $b = \sup\{x \in U \mid x < c\}$. Then b exists and $a \le b \le c$. Since S is an interval, $b \in S$, so $b \in U$ or $b \in V$. If $b \in U$ then $b < c$ ($c \in V$ and $V \cap U = \emptyset$). But by openness of U in S, $(b_-, b_+) \cap S \subseteq U$ for some $b_- < b < b_+$. Picking b' so that $b < b' < \min\{b_+, c\}$ we have $[b, b'] \subseteq U$ for some $b' > b$, contradicting the definition of b (since $\sup \ge b'$). If $b \in V$ then $b > a$ ($a \in U$ and $U \cap V = \emptyset$). But by openness of V in S, $(b_-, b_+) \cap S \subseteq V$ for some $b_- < b < b_+$. Picking b' so that $\max\{b_-, a\} < b' < b$ we have $[b', b] \subseteq V$ for some $b' < b$, contradicting the definition of b ($\sup \le b'$). □

Intermediate Value Theorem *Assume X is connected and $f\colon X \to \mathbb{R}$ is continuous. If $f(x) < t < f(y)$ then there exists a $z \in X$ with $f(z) = t$.*

Proof. $f[X]$ is an interval. □

Example The ordered square $[0, 1] \times [0, 1]$ is connected but not path connected. The image of a path from $(0, 0)$ to $(1, 1)$ is connected, so is an interval, so is the whole of $[0, 1] \times [0, 1]$, but $[0, 1] \times [0, 1]$ contains uncountably many disjoint open intervals $\{x\} \times (0, 1)$ whose inverse image would give uncountably many disjoint open sets in $[0, 1]$, each of which would contain a rational. The long line $\Omega \times [0, 1)$ is connected and actually path connected, but not homeomophic to $\mathbb{R}$ for similar reasons.

Components

If $x, y \in X$, we say x and y are *(path) connected* in X if there exists a (path) connected subspace containing x and y. By Lemma 4.3 these are equivalence relations on X. The equivalence classes are called the (path) components of X. Equivalently, the (path) components are maximal (path) connected subsets of X. Components are closed (Lemma 4.6) but path components need not be. A space is called *totally (path) disconnected* if all (path) components are singletons.

A space $(X, \mathcal{T})$ is *locally (path) connected* if it has a base consisting entirely of (path) connected sets. Note that connected $\not\Rightarrow$ locally connected (topologist's sine curve) and locally connected $\not\Rightarrow$ connected (two disjoint intervals).

Lemma 4.9 *Locally path connected + connected ⇒ path connected.*

Proof. Let C be a path component in X and let $x \in C$. Then there exists a path connected open $U \ni x$ by local path connectivity. Since $C \cup U$ is path connected and C is a path component, $C \cup U = C$ and $x \in U \subseteq C$. Since this holds for all $x \in C$, C is open. But $C = X \setminus \bigcup C'$ where the union is over all other path components in X. Since $\bigcup C'$ is open, C is also closed. By connectivity $C = X$ and X is path connected. □

Corollary *For open subsets of $\mathbb{R}^n$, connected ⇔ path connected.*

Proof. Any open subset of $\mathbb{R}^n$ is locally path connected. □

Note that this fails for closed subsets of $\mathbb{R}^n$, e.g., the topologist's sine curve.

Math 7411 5. Compactness Spring 2008

Definition A topology $(X, \mathcal{T})$ is *compact* if every collection of open sets $\mathcal{U} \subseteq \mathcal{T}$ that covers X, i.e., $\bigcup_{U \in \mathcal{U}} U = X$, has some finite subcover: $\{U_1, \dots, U_n\} \subseteq \mathcal{U}$, $\bigcup_{i=1}^n U_i = X$.

Equivalently, X is compact iff whenever $\mathcal{C}$ is a collection of closed sets such that every finite intersection $\bigcap_{i=1}^n C_i$, $C_i \in \mathcal{C}$, is non-empty, then $\bigcap_{C \in \mathcal{C}} C$ is non-empty.
[Often, and equivalently, $\mathcal{C}$ is a nested collection of closed sets, i.e., totally ordered by $\subseteq$.]

We say a subset $K \subseteq X$ is compact if it compact in the subspace topology. Equivalently K is compact if whenever $\mathcal{U}$ is a collection of open sets (in X) with $K \subseteq \bigcup_{U \in \mathcal{U}} U$ there is a finite subcollection $\{U_1, \dots, U_n\} \subseteq \mathcal{U}$ with $K \subseteq \bigcup_{i=1}^n U_i$.

Lemma 5.1 *If $(X, \leq)$ satisfies the LUB axiom and is given the order topology, then any interval of the form $[a, b]$ is compact in X.*

Proof. Let $[a, b] \subseteq \bigcup_{U \in \mathcal{U}} U$. Let $S = \{c \leq b \mid [a, c) \text{ is covered by a finite subcollection of } \mathcal{U}\}$. Clearly $a \in S$ and b is an upper bound of S. Let $c_0 = \sup S$. Since $c_0 \in [a, b] \subseteq \bigcup_{U \in \mathcal{U}} U$, $\exists U_0 \in \mathcal{U} \colon c_0 \in U_0$. Since U_0 is open, $c_0 \in (c_-, c_+) \subseteq U_0$ where $c_- < c_0 < c_+$. By definition of c_0, c_- is not an upper bound for S, and so $\exists c' > c_-, U_1, \dots, U_n \in \mathcal{U} \colon [a, c') \subseteq \bigcup_{i=1}^n U_i$. But then $[a, c_+) \subseteq [a, c') \cup (c_-, c_+) \subseteq \bigcup_{i=0}^n U_i$. Thus either $c_+ > b$ or $c_+ \in S$. But $c_+ > c_0 = \sup S$, so $c_+ \notin S$ and so $c_+ > b$ and $[a, b] \subseteq [a, c_+) \subseteq \bigcup_{i=0}^n U_i$. □

Lemma 5.2 *Any closed subspace of a compact space is compact.*

Proof. If $\mathcal{U}$ is an open cover of C and C is a closed subspace of X then $\mathcal{U} \cup \{X \setminus C\}$ is an open cover of X. A finite subcover of this covers X, and on removing $X \setminus C$ if necessary we obtain a finite subcover of C. □

Heine-Borel Theorem *A subset of $\mathbb{R}$ is compact iff it is closed and bounded.*

Proof. If S is closed and bounded then it is a closed subspace of $[a, b]$ for some $a, b \in \mathbb{R}$. Hence it is compact. Conversely, if S is not bounded then $\{(-n, n) \mid n \in \mathbb{N}\}$ is an open cover of S with no finite subcover. If S is not closed, pick $x \in \bar{S}$, $x \notin S$. Then $\{(-\infty, x - \varepsilon) \cup (x + \varepsilon, \infty) \mid \varepsilon > 0\}$ is an open cover of S with no finite subcover. □

Lemma 5.3 *Any continuous image of a compact space is compact.*

Proof. If $f \colon X \to Y$ is continuous and $\mathcal{U}$ is an open cover of $f[Y]$, then $\{f^{-1}[U] \mid U \in \mathcal{U}\}$ is an open cover of X. A finite subcover $\{f^{-1}[U_i] \mid i = 1, \dots, n\}$ gives a finite subcover $\{U_1, \dots, U_n\}$ of $f[X]$. □

Corollary *Any continuous function $f \colon X \to \mathbb{R}$ from a non-empty compact space X is bounded and attains its bounds.*

Proof. $f[X]$ is a non-empty compact subset of $\mathbb{R}$. □

Lemma 5.4 *If $K_1, \dots, K_n$ are compact subsets of X then the finite union $K_1 \cup \cdots \cup K_n$ is compact.*

Proof. Take an open cover of the union. Then it is an open cover of each K_i individually. The union of the finite subcovers for each K_i gives a finite subcover for the union. □

Alexander's Subbase Theorem *Suppose $\mathcal{S}$ is a subbase for the topology $(X, \mathcal{T})$. Then X is compact iff every collection of subbasis sets $\{S_\alpha \in \mathcal{S} : \alpha \in I\}$ that covers X has a finite subcover $\{S_{\alpha_1}, \dots, S_{\alpha_n}\}$.*

Proof. $\Rightarrow$: Clear since every subbasis set is open.
$\Leftarrow$: Let $\mathcal{U}$ be a collection of open sets with no finite subcover. It is enough to prove that $\mathcal{U}$ does not cover X.

Let $\mathcal{X} = \{\mathcal{C} \mid \mathcal{U} \subseteq \mathcal{C} \subseteq \mathcal{T}$ and $\mathcal{C}$ has no finite subcover$\}$. Clearly $\mathcal{U} \in \mathcal{X}$, so $\mathcal{X} \neq \emptyset$. Order $\mathcal{X}$ by inclusion, i.e., $\mathcal{C} \leq \mathcal{C}'$ iff $\mathcal{C} \subseteq \mathcal{C}'$. Consider a chain $\{\mathcal{C}_\beta\}$ in $\mathcal{X}$ and let $\mathcal{C} = \bigcup \mathcal{C}_\beta$. Then $\mathcal{U} \subseteq \mathcal{C} \subseteq \mathcal{T}$. Consider a finite subcollection $\{U_1, \dots, U_n\} \subseteq \mathcal{C}$. Then for each i, $U_i \in \mathcal{C}_{\beta_i}$ for some β_i. Taking the largest such $\mathcal{C}_{\beta_i}$ in our chain $\mathcal{C}_{\beta_{i_0}}$, we get $\{U_1, \dots, U_n\} \subseteq \mathcal{C}_{\beta_{i_0}} \in \mathcal{X}$ so $\{U_1, \dots, U_n\}$ does not cover X and so $\mathcal{C} \in \mathcal{X}$. Clearly $\mathcal{C}$ is an upper bound for the chain. Applying Zorn's Lemma we obtain a maximal element $\mathcal{M} \in \mathcal{X}$, i.e., a collection of open sets with no finite subcover, but adding any single additional open U to $\mathcal{M}$ gives rise to a collection $\{U\} \cup \mathcal{M}$ with a finite subcover of X.

Now consider $\mathcal{M}' = \mathcal{M} \cap \mathcal{S}$, i.e., remove any set from $\mathcal{M}$ that is not in the subbase. Clearly $\mathcal{M}'$ has no finite subcover. Thus by assumption $\mathcal{M}'$ does not cover X. Let $x \in X$ be an element not covered by $\mathcal{M}'$. I claim it is also not covered by $\mathcal{M}$ and hence not covered by $\mathcal{U} \subseteq \mathcal{M}$. Assume otherwise and suppose $x \in U \in \mathcal{M}$. Since $\mathcal{S}$ is a subbase, we can write $x \in S_1 \cap \cdots \cap S_n \subseteq U$ where $S_i \in \mathcal{S}$. Moreover, $S_i \notin \mathcal{M}$ since otherwise $S_i \in \mathcal{M}'$ and x would be covered by $\mathcal{M}'$. By maximality of $\mathcal{M}$, there is a finite subcollection of $\{S_i\} \cup \mathcal{M}$ that covers X, say $\{S_i, U_{i,1}, \dots, U_{i,k_i}\}$. But then

$$X = \bigcap_i \Big(S_i \cup \bigcup_j U_{i,j}\Big) \subseteq (S_1 \cap \cdots \cap S_n) \cup \bigcup_{i,j} U_{i,j} \subseteq U \cup \bigcup_{i,j} U_{i,j},$$

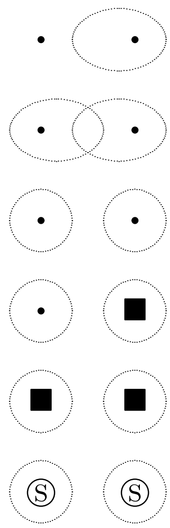
contradicting the assumption that $\mathcal{M}$ has no finite subcover. □

Tychonoff's Theorem *A product of compact spaces is compact.*

Proof. By Alexander's Theorem, it is enough to show that any cover by simple cylinder sets (i.e., sets of the form $U_{\alpha_0} \times \prod_{\beta \neq \alpha_0} X_\beta$) has a finite subcover. Suppose $\mathcal{U}$ is a collection of simple cylinder sets with no finite subcover and write $\mathcal{U} = \bigcup \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha$ are those simple cylinder sets of $\mathcal{U}$ with $\alpha_0 = \alpha$. If some $\mathcal{U}_\alpha = \{U_{\alpha,i} \times \prod_{\beta \neq \alpha} X_\beta \mid i \in I\}$ covers X, then $\{U_{\alpha,i} \mid i \in I\}$ would cover X_α. Compactness of X_α would then give us a finite subcover of X_α, which would correspond to a finite subcover in $\mathcal{U}_\alpha$ of $\prod X_\alpha$. Hence we may assume there is some $x_\alpha \in X_\alpha$ that is not covered by any $U_{\alpha,i}$. Then any $y \in \prod X_\alpha$ with α-coordinate $y_\alpha = x_\alpha$ is not covered by $\mathcal{U}_\alpha$, since whether or not y is covered by $\mathcal{U}_\alpha$ depends only on the α-coordinate. Thus the element $x = (x_\alpha) \in \prod X_\alpha$ is not covered by $\mathcal{U}_\alpha$ for any α, and so is not covered by $\mathcal{U}$. □

Math 7411 — 6. Separation Axioms — Spring 2008

Whereas compactness and connectedness say that there are not "too many" open sets, the following properties all say that there are "enough" open sets in the topology.



T_0 (**Kolmogorov**) $\forall x, y \colon x \neq y \Rightarrow$
$\exists$ open $U \colon x \in U, y \notin U$ **or** $\exists$ open $V \colon x \notin V, y \in V$

T_1 (**Fréchet**) $\forall x, y \colon x \neq y \Rightarrow$
$\exists$ open $U \colon x \in U,\ y \notin U$ **and** $\exists$ open $V \colon x \notin V,\ y \in V$

T_2 (**Hausdorff**) $\forall x, y \colon x \neq y \Rightarrow$
$\exists$ open $U, V \colon x \in U,\ y \in V,\ U \cap V = \emptyset$

T_3 (**Regular**) Points are closed†, and $\forall x$, closed $B \colon x \notin B \Rightarrow$
$\exists$ open $U, V \colon x \in U,\ B \subseteq V,\ U \cap V = \emptyset$

T_4 (**Normal**) Points are closed†, and $\forall$ closed $A, B \colon A \cap B = \emptyset \Rightarrow$
$\exists$ open $U, V \colon A \subseteq U,\ B \subseteq V,\ U \cap V = \emptyset$

T_5 (**Completely Normal**) Points are closed†, and $\forall$ separated* A, B:
$\exists$ open $U, V \colon A \subseteq U,\ B \subseteq V,\ U \cap V = \emptyset$

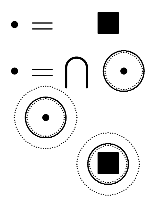
*A and B are *separated* if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Equivalently, there is some open $U \supseteq A$ disjoint from B and some open $V \supseteq B$ disjoint from A.

† Some people assume "points are closed" for the T_i but not for the "named" conditions (Regular, ...). Others assume it for the names but not the T_is! I will assume it for both.

Note: $T_5 \Rightarrow T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$ but none of the reverse implications hold!

$T_0 \not\Rightarrow T_1$: $\mathcal{T} = \{\emptyset, \{0\}, \{0,1\}\}$ on $X = \{0,1\}$. $T_1 \not\Rightarrow T_2$: Cofinite topology on infinite set. $T_2 \not\Rightarrow T_3$: Set $K = \{1/n \mid n \geq 1\}$ and give $\mathbb{R}$ basis consisting of (a,b) and $(a,b) \setminus K$ for all a, b. Use $x = \{0\}$ and $B = K$ to show not regular. $T_3 \not\Rightarrow T_4$ and $T_4 \not\Rightarrow T_5$ examples later.

Some equivalent conditions:



$T_1 \Leftrightarrow$ every point is closed.

$T_2 \Leftrightarrow$ every point is the intersection of the closures of their neighborhoods.

$T_3 \Leftrightarrow T_1 + \forall$ open $U \ni x \colon \exists$ open $V \colon U \supseteq \bar{V} \supseteq V \ni x$.

$T_4 \Leftrightarrow T_1 + \forall$ open $U \supseteq C$, C closed: $\exists$ open $V \colon U \supseteq \bar{V} \supseteq V \supseteq C$.

Of the above conditions, probably the most important is Hausdorff (T_2).

Lemma 6.1 *X is T_2 $\Leftrightarrow$ the diagonal $\Delta_X = \{(x,x) \mid x \in X\}$ is closed in $X \times X$.*

Proof. Δ_X is closed $\Leftrightarrow$ $(\forall (x,y) \notin \Delta_X \colon \exists$ open $U \times V \ni (x,y)$ disjoint from $\Delta_X)$ $\Leftrightarrow$ $(\forall x \neq y \colon \exists$ open $U \ni x,\ V \ni y \colon U \cap V = \emptyset)$ $\Leftrightarrow$ T_2. □

Lemma 6.2 *If $f, g \colon X \to Y$ are continuous and Y is Hausdorff, then $\{x \in X \mid f(x) = g(x)\}$ is closed.*

Proof. Consider the inverse image of Δ_Y under the map $x \mapsto (f(x), g(x))$. $\square$

Lemma 6.3 *In a Hausdorff space, limits of sequences are unique (if they exists), i.e., if $x_n \to x$ and $x_n \to x'$ then $x = x'$.*

Proof. Consider disjoint open $U \ni x$ and $V \ni x'$. Both contain almost all x_n. $\square$

Metric spaces satisfy all these T_i axioms:

Lemma 6.4 *If S is a subset of a metric space then $\overline{S} = \{x \mid d(x, S) = 0\}$ where we define $d(x, S) = \inf\{d(x, y) \mid y \in S\}$.*

Proof. $x \in \overline{S} \iff (\forall r > 0\colon B(x, r) \cap S \neq \emptyset) \iff (\forall r > 0\colon d(x, S) < r) \iff d(x, S) = 0$. $\square$

Lemma 6.5 *Any metric space is completely normal (normal, regular, Hausdorff, ...).*

Proof. Let A, B be separated sets and let $U = \{x \in X \mid d(x, A) < d(x, B)\}$, $V = \{x \in X \mid d(x, B) < d(x, A)\}$. Clearly $U \cap V = \emptyset$. If $x \in A$ then $x \notin \overline{B}$, so $d(x, B) > 0$ and $d(x, A) = 0$, so $x \in U$. Thus $U \supseteq A$, and similarly $V \supseteq B$. If $x \in U$ then $d(x, B) - d(x, A) = \varepsilon > 0$. By the triangle inequality, if $y \in B(x, \varepsilon/3)$ then $d(y, B) - d(y, A) \geq \varepsilon - \varepsilon/3 - \varepsilon/3 > 0$, so $y \in U$. Thus U (and similarly V) is open. Since any metric space is T_2 (if $x \neq y$, $r = d(x, y)$, then $B(x, r/2) \ni x$ and $B(y, r/2) \ni y$ are disjoint), it is T_1 and hence T_5. $\square$

Order topologies also satisfy all these T_i axioms, however we shall not prove that here.

Lemma 6.6 *A subspace of a T_i space is T_i for $i \neq 4$.*

Proof. Exercise. $\square$

A subspace of a T_4 space need not be T_4 (I will give an example later). However,

Lemma 6.7 *A space is T_5 iff all subspaces are T_4.*

Proof. $\Rightarrow$ clear from Lemma 6.6. For $\Leftarrow$, use T_4 applied to the disjoint closed subsets $\overline{A} \cap S$ and $\overline{B} \cap S$ in the subspace $S = X \setminus (\overline{A} \cap \overline{B})$. $\square$

Lemma 6.8 *A product of T_i spaces is T_i for $i = 0, 1, 2, 3$.*

Proof. (For T_3, others easier.) Let $x = (x_\alpha) \in X = \prod X_\alpha$ and $C \subseteq X$ a closed set $x \notin C$. Now there exists a basic open set $W = \prod W_\alpha$ such that $x \in W \in X \setminus C$. For each α such that $W_\alpha \neq X_\alpha$ pick disjoint $U_\alpha \ni x_\alpha$ and $V_\alpha \supseteq X_\alpha \setminus W_\alpha$. For all other α, $U_\alpha = V_\alpha = X_\alpha$. Then $U = \prod U_\alpha$ and $V = \bigcup_\alpha (V_\alpha \times \prod_{\beta \neq \alpha} X_\beta)$ are as required. Finally if we assume the result for T_1, T_1 for each X_α implies T_1 for X, so points are closed. $\square$

A product of T_5 spaces need not even be T_4 (example later).

None of the T_i axioms are preserved under quotients in general. All are preserved under disjoint unions of topological spaces.

Math 7411 7. Compact Hausdorff Spring 2008

Lemma 7.1 *Any compact subset $S \subseteq X$ of a Hausdorff space X must be a closed subset.*

Recall: Every closed subset of a compact space is compact, so in a compact Hausdorff space, subsets are compact iff they are closed.

Proof. If $x \notin S$, then for each $y \in S$ there are open $U_y \ni x$, $V_y \ni y$ with $U_y \cap V_y = \emptyset$. The collection $\{V_y \mid y \in S\}$ is a cover. If is had a finite subcover $\{V_{y_i} \mid i = 1, \ldots, n\}$ then $U = \bigcap_{i=1}^n U_{y_i}$ is open, contains x, but does not intersect S. Thus $x \notin \overline{S}$. □

Lemma 7.2 *If $f\colon X \to Y$ is a continuous bijective function from a compact space X to a Hausdorff space Y, then f is a homeomorphism.*

Proof. If C is closed in X, then it is compact, so $(f^{-1})^{-1}[C] = f[C]$ is compact in Y, so is closed. Thus f^{-1} is continuous. □

Lemma 7.3 *If K and K' are disjoint compact subsets of a Hausdorff space then there exist open $U \supseteq K$ and $V \supseteq K'$ with $U \cap V = \emptyset$. In particular, a compact Hausdorff space is Normal.*

Proof. For each $x \in K$, $y \in K'$, let $U_{x,y} \ni x$ and $V_{x,y} \ni y$ be disjoint open sets. For each fixed x, $\{V_{x,y} \mid y \in K'\}$ covers K', so has a finite subcover $\{V_{x,y_i} \mid i = 1, \ldots, n\}$. Then $U_x = \bigcap_{i=1}^n U_{x,y_i}$ is open, contains x, and is disjoint from $V_x = \bigcup_{i=1}^n V_{x,y_i}$, which is open and contains K'. Now $\{U_x \mid x \in K\}$ is a cover of K, so has a finite subcover $\{U_{x_j} \mid j = 1, \ldots, m\}$. But then $U = \bigcup_{j=1}^m U_{x_j}$ and $V = \bigcap_{j=1}^m V_{x_j}$ are as desired. Finally, every closed set in a compact space is compact, so for a compact Hausdorff space we have proved T_4 (Hausdorff $\Rightarrow$ points are closed). □

Example Let Ω be the first uncountable ordinal, and ω the first infinite ordinal. Then $[0, \Omega]$ (the set of ordinals from 0 to Ω with the order topology) and $[0, \omega]$ are both compact (Lemma 5.1), T_2 (order topology). Thus $X = [0, \Omega] \times [0, \omega]$ is compact, T_2, so T_4. Consider $A = [0, \Omega) \times \{\omega\}$ and $B = \{\Omega\} \times [0, \omega)$. One can check that if $V \supseteq B$ is open then $(\alpha, \Omega] \times [0, \omega) \subseteq V$ for some countable α. Then $(\beta, \omega) \in A$ has no open neighborhood disjoint from V for any $\alpha < \beta < \Omega$. A and B are separated in X and closed in $S = X \setminus \{(\Omega, \omega)\}$. Thus X is T_4 but not T_5, S is T_3 ($\subseteq T_3$) but not T_4. S is also a non-T_4 subspace of a T_4 space, and X shows that Compact + $T_2 \not\Rightarrow T_5$.

Definition A subset S of a topological space X is

dense if $\overline{S} = X$, equivalently $S \cap U \neq \emptyset$ for any non-empty open set U;

nowhere dense if $\overline{S}$ has empty interior, equivalently $X \setminus \overline{S}$ is dense;

1st category if S is a countable union of nowhere dense sets;

2nd category if it is not 1st category.

Definition X is a *Baire* space if whenever U_i, $i \in \mathbb{N}$, is a countable collection of dense open sets, $\bigcap_{i=0}^{\infty} U_i$ is dense.

Equivalently: every set with non-empty interior is of 2nd category.

Proof. $\Leftarrow$: $X \setminus U_i$ is nowhere dense, so $X \setminus \bigcap U_i = \bigcup (X \setminus U_i)$ has empty interior, so $\bigcap U_i$ is dense. $\Rightarrow$: If S_i are nowhere dense then $X \setminus \overline{S}_i$ are dense and open. Then $X \setminus \bigcup \overline{S}_i = \bigcap (X \setminus \overline{S}_i)$ is dense, so $\bigcup S_i \subseteq \bigcup \overline{S}_i$ has empty interior. □

Baire Category Theorem Every compact Hausdorff space (or complete metric space) is Baire.

Proof. Suppose V_i is a nonempty open set. Then $V_i \cap U_i$ is an open set that is non-empty since U_i is dense. Pick $x_i \in V_i \cap U_i$ and choose an open $V_{i+1} \ni x_i$ with $\overline{V}_{i+1} \subseteq U_i \cap V_i$ (by regularity). Hence starting with an arbitrary nonempty open set V_0, we can construct inductively V_i with $\overline{V}_{i+1} \subseteq U_i \cap V_i$. Then any finite intersection $\bigcap_{i \in I} \overline{V}_i = \overline{V}_{\max(I)} \neq \emptyset$, so by compactness $\emptyset \neq \bigcap_{i=0}^{\infty} \overline{V}_{i+1} \subseteq V_0 \cap \bigcap_{i=0}^{\infty} U_i$. Since V_0 is arbitrary, $\bigcap_{i=0}^{\infty} U_i$ is dense.

For complete metric spaces we follow the same proof as above, with the extra condition that when we choose $V_{i+1} \ni x_i$, we ensure that $V_{i+1} \subseteq B(x_i, 2^{-i})$. Now $x_j \in V_j \subseteq V_i$ for all $j > i$, so $d(x_j, x_i) < 2^{-i}$ for all $j > i$. Thus x_i is a Cauchy sequence, and the limit $x = \lim x_j$ lies in every $\overline{V}_i$. Thus $\bigcap \overline{V}_i \neq \emptyset$ (without the need for compactness). The rest of the proof is the same as before. □

The assumptions in the Baire category theorem can be weakened. The proof only requires countably compact + regular. (*Countably compact*: every countable open cover has a finite subcover). Alternatively we can use locally compact + Hausdorff. (*Locally compact*: every $x \in X$ is in the interior of some compact subspace of X.) Or even locally countably compact + regular. (*Locally countably compact*: ... guess the definition!).

Corollary *Any non-empty compact Hausdorff (or complete metric space, or ...) without isolated points is uncountable.*

Proof. $\{x\}$ is closed (Hausdorff) and has empty interior (x not isolated). Thus $\{x\}$ is nowhere dense. Thus X (interior $= X \neq \emptyset$) is not a countable union of singletons. □

Example The lower limit topology on $\mathbb{R}$ is T_5 (exercise). The *Sorgenfrey* plane X is $\mathbb{R}^2$ with the product topology of two copies of the lower limit topology. Let $L = \{(x, -x) \mid x \in \mathbb{R}\}$. Check $A = \{(x, -x) \mid x \in \mathbb{Q}\}$ and $B = L \setminus A$ are closed. Assume $V \supseteq B$ is open. Let $S_n = \{x \in \mathbb{R} \mid [x, x + 1/n) \times [-x, -x + 1/n) \subseteq V\}$. Then $\bigcup_n S_n = \mathbb{R} \setminus \mathbb{Q}$, so by Baire (usual $\mathbb{R}$ topology, $\bigcup_n S_n \cup \bigcup_{q \in \mathbb{Q}} \{q\} = \mathbb{R}$) gives $(a, b) \subseteq \overline{S}_n$ for some n. Pick $q \in \mathbb{Q} \cap (a, b)$. Then every open neighborhood of $(q, -q) \in A$ intersects V. Thus X is not T_4. In particular a product of T_5 spaces is not necessarily T_4, and X is a T_3 space ($T_3 \times T_3$) that is not T_4.

Math 7411 8. Sequences and nets Spring 2008

Definition X is *1st countable* if every $x \in X$ has a countable base of neighborhoods of x.

Example Every metric space is 1st countable, since $\{B(x, 1/n) \mid n \geq 1\}$ is a countable base of neighborhoods of x.

1st countability allows one to define topological properties, such as the closure of a set or continuity of functions, in terms of sequences:

Theorem 8.1 *In any space, if $x_n \to x$ and $x_n \in A$ then $x \in \overline{A}$. In a 1st countable space if $x \in \overline{A}$ then there exists a sequence $x_n \in A$ with $x_n \to x$.*

Proof. If $x_n \to x$, $x_n \in A$, then any open $U \ni x$ contains some (most) x_n, so contains points of A. Thus $x \in \overline{A}$. If there is a countable base of neighborhoods of $x \in \overline{A}$, say $\mathcal{B}_x = \{B_1, B_2, \dots\}$, then $U_n = B_1 \cap \cdots \cap B_n$ is a neighborhood of x. Thus $A \cap U_n \neq \emptyset$. Pick $x_n \in A \cap U_n$. Then $x_n \to x$ (since any $U \ni x$ contains some B_n, and $x_m \in U_m \subseteq B_n \subseteq U$ for all $m \geq n$). $\square$

Theorem 8.2 *If $f\colon X \to Y$ is continuous at x and $x_n \to x$ in X then $f(x_n) \to f(x)$ in Y. If X is 1st countable and $f(x_n) \to f(x)$ for all $x_n \to x$ then f is continuous at x.*

Proof. If f is continuous at x, then $\forall$ open $V \ni f(x)\colon \exists$ open $U \ni x\colon f[U] \subseteq V$. If $x_n \to x$ then $x_n \in U$ and hence $f(x_n) \in V$ for all sufficiently large n. Thus $f(x_n) \to f(x)$. Now suppose $\mathcal{B}_x = \{B_1, B_2, \dots\}$ is a countable base of neighborhoods of $x \in X$, and suppose f is not continuous at x. Let $U_n = B_1 \cap \cdots \cap B_n$. Then $\exists$ open $V \ni f(x)\colon \forall n\colon f[U_n] \not\subseteq V$. Pick $x_n \in U_n$ with $f(x_n) \notin V$. Then $x_n \to x$ but $f(x_n) \not\to f(x)$. $\square$

Example Let Ω be the first uncountable ordinal, $X = [0, \Omega]$, and $A = [0, \Omega)$. Then $\Omega \in \overline{A}$, but there is no sequence in A tending to Ω. Similarly, we can define $f\colon X \to \{0, 1\}$ by sending A to 0 and Ω to 1. This is not continuous at Ω, but if $x_n \to \Omega$ then $x_n = \Omega$ for all $n \geq n_0$, and so $f(x_n) \to f(\Omega)$.

Definition A *directed set* is a partially ordered set I such that $\forall \alpha, \beta \in I\colon \exists \gamma \in I\colon \gamma \geq \alpha, \beta$. Note that any totally ordered set is a directed set.

Definition A *net* in X, written $(x_\alpha)_{\alpha \in I}$, is a function from a directed set I to X.
(x_α) *converges* to x, or $x = \lim x_\alpha$, if $\forall$ open $U \ni x\colon \exists \alpha \in I\colon \forall \gamma \geq \alpha\colon x_\gamma \in U$.
x is an *accumulation point* of $(x_\alpha)_{\alpha \in I}$ if $\forall$ open $U \ni x\colon \forall \alpha \in I\colon \exists \gamma \geq \alpha\colon x_\gamma \in U$.

If $I = \mathbb{N}$ with the usual order, then this is simply the usual convergence of a sequence.

Theorem 8.3 *In any space, $x \in \overline{A}$ iff there exists a net $(x_\alpha)_{\alpha \in I}$ with $x_\alpha \in A$ and $x_\alpha \to x$.*

Proof. $\Leftarrow$ as for Theorem 8.1. $\Rightarrow$: Let I be the set of neighborhoods of x ordered by reverse inclusion and choose $x_U \in U \cap A$. $\square$

Theorem 8.4 *In any space $f: X \to Y$ is continuous at x iff for every net $(x_\alpha)_{\alpha\in I}$ converging to x, the net $(f(x_\alpha))_{\alpha\in I}$ converges to $f(x)$.*

Proof. $\Rightarrow$ as for Theorem 8.3. $\Leftarrow$: Let I be the set of neighborhoods of x ordered by reverse inclusion and suppose there is no open $U \ni x$ with $f[U] \subseteq V$ where V is a neighborhood of $f(x)$. Pick $x_U \in U$ with $f(x_U) \notin V$. Then $x_U \to x$ but $f(x_U) \not\to f(x)$. □

Definition A *subnet* of a net $(x_\alpha)_{\alpha\in I}$ is the composition $(x_{h(\beta)})_{\beta\in J}$ where $h: J \to I$ is monotone ($\beta \le \beta' \Rightarrow h(\beta) \le h(\beta')$) and cofinal ($\forall\alpha \in I: \exists\beta \in J: h(\beta) \ge \alpha$).

Note: A sequence is a net (with $I = \mathbb{N}$), and a subsequence of a sequence is a subnet of that sequence. However, not all subnets of a sequence are subsequences.

Lemma 8.5 *A point is an accumulation point of a net iff it is a limit of some subnet.*

Proof. $\Leftarrow$: clear. $\Rightarrow$: Let x be an accumulation point and let J be the set of pairs (α, U) where U is open and $x, x_\alpha \in U$. Define $(\alpha, U) \le (\beta, V)$ if $\alpha \le \beta$ and $U \supseteq V$. define $h: J \to I$ by $h(\alpha, U) = \alpha$. Then x is a limit of the subnet $(x_{h(\alpha,U)})_{(\alpha,U)\in J}$. □

Theorem 8.6 *A space X is compact iff every net has a convergent subnet.*

Proof. $\Rightarrow$: Let $(x_\alpha)_{\alpha\in I}$ be any net and let $T_\alpha = \{x_\beta \mid \beta \ge \alpha\}$. Then $\overline{T}_\alpha$ is a collection of closed sets, no finite intersection of which is empty (contains x_γ, where $\gamma \ge$ all the α's). Thus $\bigcap \overline{T}_\alpha \ne \emptyset$. Pick $x \in \bigcap_\alpha \overline{T}_\alpha$. Then any open $U \ni x$ intersects every T_α, so $\forall\alpha \in I: \exists\gamma \ge \alpha: x_\gamma \in U$. Hence x is an accumulation point, so is a limit of some subnet. $\Leftarrow$: Let $\{C_\alpha \mid \alpha \in I\}$ be a collection of closed sets, no finite intersection being empty. Let J be the set of finite subsets of I ordered by inclusion. Pick $x_S \in \bigcap_{\alpha\in S} C_\alpha$ for each $S \in J$. Let x be an accumulation point of $(x_S)_{S\in J}$. then for any $U \ni x$ and $\alpha \in I$, there is an $S \supseteq \{\alpha\}$ with $x_S \in U$. Hence $x_S \in U \cap C_\alpha$, so $U \cap C_\alpha \ne \emptyset$. Since this holds for all U, $x \in \overline{C}_\alpha = C_\alpha$ for all α, and so $\bigcap_\alpha C_\alpha \ne \emptyset$. Thus X is compact. □

Definition X is *sequentially compact* if every sequence $(x_n)_{n=1}^\infty$ in X has a convergent subsequence $(x_{n_k})_{k=1}^\infty$.

The Bolzano Weierstrass Theorem of real analysis states that every closed bounded interval in $\mathbb{R}$ is sequentially compact, which is the same condition needed for compactness. This is no coincidence since for metric spaces they are equivalent (see next section). However they are not equivalent in general, even assuming 1st countability.

Examples $[0, \Omega)$ is sequentially compact (consider the subsequence of terms x_n where $x_n = \min\{x_m \mid m \ge n\}$) but is not compact. It is however 1st countable.
$\prod_{x\in[0,1]}[0, 1]$ (the set of functions $[0, 1] \to [0, 1]$ with "pointwise convergence" topology) is compact but not sequentially compact. (Let $f_n(x) =$ the n'th digit a_n in the binary expansion $x = \sum a_n 2^{-n}$. Given any subsequence f_{n_k} consider x_0 with n_k'th binary digit $= 0$ if k even and $= 1$ if k odd. Then $f_{n_k}(x_0)$ does not converge, so neither does f_{n_k}.)

Math 7411 9. Countability Spring 2008

Recall that a topological space $(X, \mathcal{T})$ is

1st countable	if every $x \in X$ has a countable base $\mathcal{B}_x$ of neighborhoods of x;
Countably compact	if every countable open cover has a finite subcover;
Sequentially compact	if every sequence has a convergent subsequence.

Theorem 9.1 *Sequentially compact $\Rightarrow$ Countably compact.*

Proof. Let $\{C_n \mid n \in \mathbb{N}\}$ be a countable collection of closed sets with no finite intersection empty. Pick $x_n \in C_0 \cap \dots \cap C_n$. Let x be a limit of some subsequence of x_n. Fix $m \in \mathbb{N}$. Then for any open $U \ni x$, U contains infinitely many x_n (all x_n of the subsequence beyond some point), so in particular contains some x_n, $n > m$. Thus U contains a point of C_m. Since this holds for all U, $x \in \overline{C}_m = C_m$. Since this holds for all m, $\bigcap C_n \neq \emptyset$. □

Theorem 9.2 *1st countable + Countably compact $\Rightarrow$ Sequentially compact*

Proof. Let x_n be any sequence and let $T_n = \{x_n, x_{n+1}, \dots\}$. Then $\overline{T}_n$ is a countable collection of closed sets, no finite intersection of which is empty. Thus $\bigcap \overline{T}_n \neq \emptyset$. Pick $x \in \bigcap \overline{T}_n$. Pick a countable base $\mathcal{B}_x = \{B_1, \dots\}$ of neighborhoods of x and let $U_n = B_1 \cap \dots \cap B_n$. Then $x \in \overline{T}_{n_{k-1}+1}$, so $U_k \cap T_{n_{k-1}+1} \neq \emptyset$. Choose $n_k > n_{k-1}$ so that $x_{n_k} \in U_k \cap T_k$. Then $x_{n_k} \to x$. □

A topological space $(X, \mathcal{T})$ is

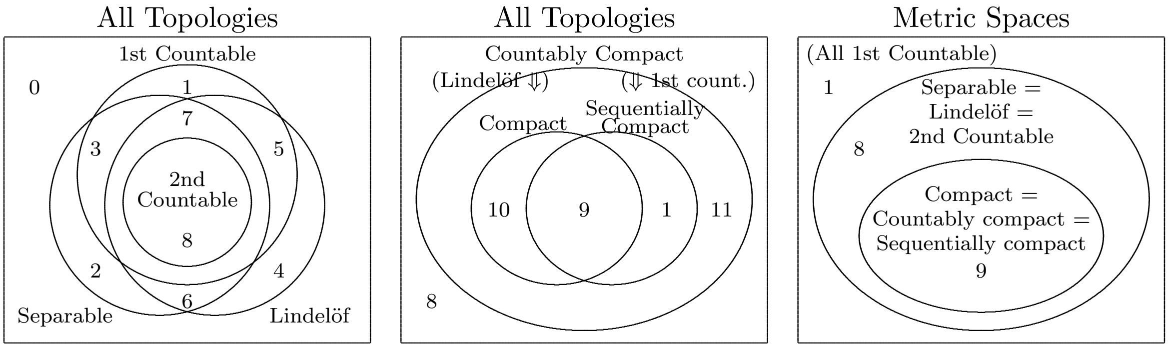
Metrizable	if the topology is induced from some metric (X, d);
2nd countable	if it has a countable base $\mathcal{B}$ for the topology;
Separable	if it has a countable dense subset;
Lindelöf	if every open cover has a countable subcover.

We have the following implications:

Metrizable $\Rightarrow$ 1st countable	[Let $\mathcal{B}_x = \{B(x, 1/n) \mid n \geq 1\}$]
2nd countable $\Rightarrow$ 1st countable	[Let $\mathcal{B}_x = \{B \in \mathcal{B} \mid x \in B\}$]
2nd countable $\Rightarrow$ Separable	[Pick one point from each $B \in \mathcal{B}$]
2nd countable $\Rightarrow$ Lindelöf	[For each $B \in \mathcal{B}$, pick if possible one $U \supseteq B$ from cover]
Compact $\Leftrightarrow$ Countably compact + Lindelöf	[Clear]

Like Compactness, Sequentially compact, Countably compact, and Lindelöf are preserved under continuous images and taking closed subspaces.

Metrizable, 1st and 2nd countable are preserved under taking subspaces, and Separable is preserved under taking *open* (but not general) subspaces.



0:$[0, \Omega 2)$, order topology.
1:$[0, \Omega)$, order topology.
2:Disjoint union of 3 and 6.
3:$\mathbb{R}$, $\mathcal{T} = \{U \mid 0 \in U$ or $U = \emptyset\}$.
4:$[0, \Omega]$, order topology.
5:$\mathbb{R}$, $\mathcal{T} = \{U \mid 0 \notin U$ or $U = \mathbb{R}\}$.
6:Co-finite topology on $\mathbb{R}$.
7:Lower limit topology on $\mathbb{R}$.
8:Standard topology on $\mathbb{R}$.
9:Standard topology on $[0, 1]$.
10:$\prod_{x \in [0,1]} [0, 1]$.
11:Disj. union of 1 and 10.

Theorem 9.3 *For metric spaces: 2nd countable ⇔ Lindelöf ⇔ Separable*

Proof. 2nd Countable ⇒ Lindelöf: always holds.
Metric + Lindelöf ⇒ Separable: For each n, $\{B(x, 1/n) \mid x \in X\}$ covers X, so take a countable subcover $\{B(x, 1/n) \mid x \in S_n\}$. Then the countable set $\bigcup S_n$ is dense.
Metric + Separable ⇒ 2nd countable: If S is a countable dense set then $\mathcal{B} = \{B(x, 1/n) \mid x \in S,\ n \geq 1\}$ is a countable basis. □

Theorem 9.4 *For metric spaces: Compact ⇔ Countably compact ⇔ Seq. compact.*

Proof. Since Metrizable ⇒ 1st countable, Theorems 10.1 and 10.2 give the second equivalence, and Compact ⇒ Countably compact is clear, so we only need Sequentially compact + Countably compact ⇒ Compact.
Sequentially compact ⇒ Separable: We show that for any n, X can be covered by countably many balls $B(x, 1/n)$. Suppose not. Then choose inductively an x_n that does not lie in $\bigcup_{i=0}^{n-1} B(x_i, 1/n)$. Let x be a limit of a subsequence of x_n. Then there must be infinitely many x_n in $B(x, 1/2n)$, but then some $x_m \in B(x_n, 1/n)$ for some $m > n$, a contradiction.
Let $\{B(x, 1/n) \mid x \in S_n\}$ cover X where S_n is countable. Then $\bigcup S_n$ is a countable dense set, so X is separable.
By Theorem 10.3, Separable ⇒ Lindelöf, but Lindelöf + Count. compact ⇒ Compact. □

Products: Metrizable, Sequentially compact, Separable, and 1st and 2nd countable are all preserved under *countable* products only, but Lindelöf and Countably compact are not preserved in general under even finite products. (For Lindelöf, the Sorgenfrey plane is a counterexample.)

Lemma 10.1 *T_3 + Lindelöf ⇒ T_4; T_3 + 2nd countable ⇒ T_5.*

Proof. $T_3 \Rightarrow$ Points are closed, so enough to show that disjoint closed sets A and B can be separated by open sets. For each $x \in A$, $T_3 \Rightarrow \exists$ open $U_x \ni x\colon \overline{U}_x \cap B = \emptyset$. Lindelöf ⇒ A is covered by countably many U_x, say $U_{x_1}, U_{x_2}, \dots$. Similarly construct V_y, $y \in B$, and $V_{y_1}, V_{y_2}, \dots$ covering B. Now $V = \bigcup_{n=1}^{\infty}(V_{y_n} \setminus \bigcup_{i=1}^{n} \overline{U}_{x_i})$ and $U = \bigcup_{n=1}^{\infty}(U_{x_n} \setminus \bigcup_{i=1}^{n} \overline{V}_{y_i})$ are as required. For the second part, 2nd Countable ⇒ Lindelöf, but T_3 and 2nd Countable are preserved under taking subspaces, so every subspace is T_4 and so the space is T_5. □

Urysohn Lemma *If X is T_4 and A and B are disjoint closed sets in X, then there exists a continuous $f\colon X \to [0,1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) = 1$ for $x \in B$.*

Proof. Let $U_1 = X \setminus B$. Using T_4, $\exists$ open $U_0 \supseteq A\colon \overline{U}_0 \subseteq U_1$. Suppose for each rational of the form $p = \frac{a}{2^n}$ we have defined open U_p so that if $q = \frac{a+1}{2^n}$, $\overline{U}_p \subseteq U_q$. Let $r = \frac{p+q}{2} = \frac{2a+1}{2^{n+1}}$, and use T_4 to define (inductively in n) U_r so that $\overline{U}_p \subseteq U_r \subseteq \overline{U}_r \subseteq U_q$. Now let $f(x) = \inf(\{p \mid x \in U_p\} \cup \{1\})$. Now for dyadic rationals p, q, $f^{-1}[(-\infty, q)] = \bigcup_{r<q} U_r$ is open and $f^{-1}[(q,\infty)] = \bigcup_{r>q}(X \setminus U_r) = \bigcup_{r>q}(X \setminus \overline{U}_r)$ is open. $[X \setminus \overline{U}_r \subseteq X \setminus U_r$ and $X \setminus U_r \subseteq X \setminus \overline{U}_{r'}$ for any $r > r' > q.]$ Since the sets $(-\infty, q)$ and (q, ∞) form a subbase for the topology on $[0,1]$, f is continuous. □

Lemma 10.2 *The* Hilbert cube $\prod_{n=1}^{\infty}[0,1]$ *is metrizable and separable.*

Proof. $d((x_n),(y_n)) = (\sum_{n=1}^{\infty} |x_n - y_n|^2/n^2)^{1/2}$ gives the product topology. The set of (x_n) with $x_n \in \mathbb{Q}$ and $x_n = 0$ for all but finitely many n is a countable dense set. □

Urysohn Metrization Theorem *T_3 + 2nd Countable ⇒ Metrizable.*

Proof. Suppose X is T_3 and 2nd countable. Then X is T_4. Let $\mathcal{B} = \{B_1, B_2, \dots\}$ be a countable base. For each pair B_n, B_m with $\overline{B}_n \subseteq B_m$, there exists a function $f_{n,m}\colon X \to [0,1]$ which is 1 on at $\overline{B}_n$ and 0 outside B_m. Construct $f\colon X \to \prod_{n,m}[0,1]$ by $f(x) = (f_{n,m}(x))_{n,m}$, where the index set is the (countable) set of pairs n, m, where $\overline{B}_n \subseteq B_m$. Now f is continuous (each component is) and injective, since if $x \neq y$ then there exists B_m with $x \in B_m$ and $y \notin B_m$ (by T_1), and a B_n with $x \in B_n \subseteq \overline{B}_n \subseteq B_m$ (by T_3), so $f_{n,m}(x) = 1 \neq 0 = f_{n,m}(y)$ and $f(x) \neq f(y)$. Let $Y = f[X] \subseteq \prod_{n,m}[0,1]$. Then f is a bijective continuous map from X to Y. We shall show f^{-1} is continuous. Pick $y \in Y$ and let $x = f^{-1}(y) \in U \subseteq X$. Choose B_n, B_m with $x \in B_n \subseteq \overline{B}_n \subseteq B_m \subseteq U$. Let V be the open subset of $\prod_{n,m}[0,1]$ where the (n,m) coordinate is not zero. Then $f^{-1}[V] \subseteq B_m \subseteq U$ and $y = f(x) \in V$. Thus for every open $U \ni f^{-1}(y)$, there is an open $V \cap Y \ni y$ in Y with $f^{-1}[V \cap Y] = f^{-1}[V] \subseteq U$. Thus $f^{-1}\colon Y \to X$ is continuous at y for every $y \in Y$. Hence f is a homeomorphism of X to a subspace of the Hilbert cube. In particular $Y = f[X]$ and hence X is metrizable. □

Note all metric spaces are T_3 and the Hilbert cube is separable, so

$$T_3 + \text{2nd Countable} \Leftrightarrow \text{Separable} + \text{Metrizable} \Leftrightarrow \text{Homeomorphic to} \subseteq \text{Hilbert cube}.$$

In particular we have a complete characterization of separable metrizable spaces. The question arises as to whether we can extend this to all metrizable spaces. We quote without proof the following:

Nagata-Smirnov Metrization Theorem *$T_3 + (\exists\ \sigma$-locally finite base$) \Leftrightarrow$ Metrizable.*

A collection of sets $\mathcal{B}$ is *locally finite* if every $x \in X$ lies in some open U that intersects only a finite number of $B \in \mathcal{B}$. A collection of sets is *σ-locally finite* if it is a countable union of locally finite collections. Unfortunately the condition "σ-locally finite base" is almost impossible to check in practice. (It is quite hard to show it even holds for metric spaces.)

Recall:

Weierstrass M-test *If $f_n\colon X \to \mathbb{R}$ are continuous, $|f_n| \leq M_n$, and $\sum_{n=1}^{\infty} M_n < \infty$, then $f(x) = \sum_{i=1}^{\infty} f_n(x)$ is a continuous function $X \to \mathbb{R}$.*

Proof. By comparison with $\sum_{n=1}^{\infty} M_n$, $\sum_{n=1}^{\infty} f_n(x)$ converges, so $f(x)$ is well defined. If $\varepsilon > 0$, $\exists N\colon \sum_{n>N} M_n < \varepsilon/3$. Since f_n is continuous, $\exists$ open $U_n \ni x\colon f_n[U_n] \subseteq (f_n(x) - \frac{\varepsilon}{3N}, f_n(x) + \frac{\varepsilon}{3N})$. Let $U = \cap_{n=1}^{N} U_n$. Then if $y \in U$, $|f_n(y) - f_n(x)| < \frac{\varepsilon}{3N}$ for $n = 1, \dots, N$ and $|\sum_{n>N} f_n(y) - \sum_{n>N} f_n(x)| < 2\varepsilon/3$, so $|f(y) - f(x)| < \varepsilon$. Thus $f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon)$. Hence f is continuous at x. □

Tietze Extension Theorem *If X is T_4 and $A \subseteq X$ is closed, then any continuous $f\colon A \to \mathbb{R}$ can be extended to a continuous $\tilde{f}\colon X \to \mathbb{R}$.*

Proof. We first prove the result with $[-1, 1]$ instead of $\mathbb{R}$. Divide the interval into three pieces, $[-1, -\frac{1}{3}]$, $[-\frac{1}{3}, \frac{1}{3}]$, and $[\frac{1}{3}, 1]$. Define $B = f^{-1}[[-1, -\frac{1}{3}]]$, $C = f^{-1}[[\frac{1}{3}, 1]]$. Then B and C are closed in A and hence in X since A is closed. By the Urysohn lemma, there is a continuous $g_1\colon X \to [-\frac{1}{3}, \frac{1}{3}]$ with $g_1(x) = -\frac{1}{3}$ on B and $g(x) = \frac{1}{3}$ on C. But then $|g_1(x)| \leq \frac{1}{3}$ and $|f(x) - g_1(x)| \leq \frac{2}{3}$ for all $x \in A$. Applying the same argument to $f - g_1$ we obtain a continuous $g_2\colon X \to \mathbb{R}$ with $|g_2(x)| \leq \frac{2}{9}$ and $|f(x) - g_1(x) - g_2(x)| \leq (\frac{2}{3})^2$. Inductively define $g_n\colon X \to \mathbb{R}$ so that $|g_n(x)| \leq \frac{1}{2}(\frac{2}{3})^n$ and $|f(x) - \sum_{i=1}^{n} g_i(x)| \leq (\frac{2}{3})^n$. Then $f(x) = \sum_{n=1}^{\infty} g_n(x)$ on A, but $\tilde{f}(x) = \sum_{n=1}^{\infty} g_n(x)$ defines a continuous function on X extending f, with $|\tilde{f}(x)| \leq 1$.

Now assume $f\colon A \to (-1, 1)$. Then by the above there is an extension of f to a continuous $\tilde{f}\colon X \to [-1, 1]$. Let $B = \{x \in X \mid |\tilde{f}(x)| = 1\}$. Then B is closed and $A \cap B = \emptyset$. Let $g\colon X \to [0, 1]$ be continuous with $g = 1$ on A and $g = 0$ on B. Then $h(x) = g(x)\tilde{f}(x)$ is a continuous extension of f with $h\colon X \to (-1, 1)$.

Finally assume $f\colon A \to \mathbb{R}$. Then $\tanh f\colon A \to (-1, 1)$, so has a continuous extension to $h\colon X \to (-1, 1)$. Then $\tanh^{-1} h(x)$ is a continuous extension of f to X. □