Connectivity of addable graph classes

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A non-empty class $\mathcal{A}$ of labelled graphs is weakly addable if for each graph $G \in \mathcal{A}$ and any two distinct components of G, any graph that can be obtain by adding an edge between the two components is also in $\mathcal{A}$. For a weakly addable graph class $\mathcal{A}$, we consider a random element R_n chosen uniformly from the set of all graph in $\mathcal{A}$ on the vertex set $\{1,\ldots,n\}$. McDiarmid, Steger and Welsh conjecture [3] that the probability that R_n is connected is at least $e^{-1/2}+o(1)$ as $n \to \infty$, and showed that it is at least e^{-1} for all n. We improve the result, and show that this probability is at least $e^{-0.7983}$ for sufficiently large n. We also consider 2-addable graph classes $\mathcal{B}$ where for each graph $G \in \mathcal{B}$ and for any two distinct components of G, the graphs that can be obtained by adding at most 2 edges between the components are in $\mathcal{B}$. We show that a random element of a 2-addable graph class on n vertices is connected with probability tending to 1 as n tends to infinity.

1 Introduction

Motivated by [3] we call a non-empty class $\mathcal{A}$ of labelled graphs *weakly addable*, if for each graph G in $\mathcal{A}$, whenever u and v are vertices in distinct components of G the graph obtained from G by adding an edge joining u and v is also in $\mathcal{A}$. In [3] a weakly addable graph class is defined so that it is also closed under isomorphism, but we do not need this additional requirement. Examples of weakly addable graph classes include forests,

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planar graphs, and triangle-free graphs, or more generally any H-free or H-minor-free class of graphs for any 2-edge-connected graph H.

For a class $\mathcal{A}$ of labelled graphs, we let $\mathcal{A}_n$ denote the set of graphs in $\mathcal{A}$ on the vertex set $[n] = \{1, \ldots, n\}$. The following conjecture was stated in [3].

Conjecture 1.1. *[3] Let $\mathcal{A}$ be any weakly addable class of graphs. Suppose that $\mathcal{A}_n$ is non-empty for all sufficiently large n, and let R_n be drawn uniformly at random from $\mathcal{A}_n$. Then*

$$\liminf_{n\to\infty} \mathbb{P}[R_n \text{ is connected}] \geq \frac{1}{\sqrt{e}}.$$

Since an element R_n chosen uniformly at random from the set $\mathcal{F}_n$ of forests with n vertices satisfies $\lim_{n\to\infty} \mathbb{P}[R_n \text{ is connected }] = 1/\sqrt{e}$ [4], the lower bound in Conjecture 1.1 cannot be strengthened. It is known [2] that $\mathbb{P}[R_n \text{ is connected }] \geq 1/e$ for an element R_n drawn uniformly at random from $\mathcal{A}_n \neq \emptyset$ of an addable class $\mathcal{A}$ for any n. We strengthen this result and prove the following theorem.

Theorem 1.2. *Let $\mathcal{A}$ be any weakly addable class of graphs. Suppose that $\mathcal{A}_n$ is non-empty, and let R_n be drawn uniformly at random from $\mathcal{A}_n$. Then for sufficiently large n*

$$\mathbb{P}[R_n \text{ is connected}] \geq e^{-0.7983}.$$

We also consider an extension of the notion of addability. We say that a non-empty class $\mathcal{B}$ of labelled graphs is 2-addable, if for each graph G in $\mathcal{B}$ and for any pair (C_1, C_2) of distinct components of G, any graph obtained from G by adding at most 2 edges between C_1 and C_2 also lies in $\mathcal{B}$. Note that a 2-addable graph class is weakly addable and thus for large n the probability that an element R_n chosen uniformly at random from a 2-addable class of graph is connected is at least $e^{-0.7983}$. We will show that in fact this probability tends to 1 as n tends to infinity as stated in the next theorem.

Theorem 1.3. *Let $\mathcal{B}$ be any 2-addable class of graphs. Suppose that $\mathcal{B}_n$ is non-empty for all sufficiently large n, and let R_n be drawn uniformly at random from $\mathcal{B}_n$. Then*

$$\lim_{n\to\infty} \mathbb{P}[R_n \text{ is connected}] = 1.$$

To prove this result we consider the maximal number of proper 2-edge-cuts a graph on n vertices may have such that one partition class has size r. We show that the cycle maximizes this number and thus there are at most n such 2-edge-cuts.

2 Weakly addable graph classes

To prove Theorem 1.2, we first show that it suffices to consider graph classes in which all connected components are trees.

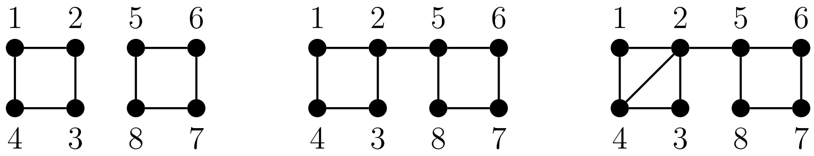

Figure 1: The first two graphs are equivalent but the third is not equivalent to the first two.

Lemma 2.1. *If for every weakly addable graph class consisting of forests only, an element R_n drawn uniformly at random from all elements of this class on $\{1,\ldots,n\}$ satisfies $\mathbb{P}[R_n \text{ is connected}] \geq x$ for some $0 \leq x \leq 1$, then an element R'_n drawn uniformly at random from all graphs $\{1,\ldots,n\}$ of any weakly addable graph class satisfies $\mathbb{P}[R'_n \text{ is connected}] \geq x$.*

Proof. Let $\mathcal{A}$ be a weakly addable graph class. We say that two graphs G, G' in $\mathcal{A}_n$ are equivalent if the graphs obtained from G and G' by removing all bridges are identical; see Figure 1. In other words, G and G' are equivalent, if they have the same 2-edge-connected blocks of size at least 3. Consider a fixed equivalence class $\mathcal{E}_n$, and the collection of 2-edge-connected blocks of size at least 3 obtained by removing all bridges from a graph $G \in \mathcal{E}_n$. Note that it does not matter which graph $G \in \mathcal{E}_n$ is chosen as all graphs in $\mathcal{E}_n$ have the same 2-edge-connected blocks of size at least 3. For each such block, we fix a tree on the same set of vertices. For each graph in the equivalence class $\mathcal{E}_n$, we replace each 2-edge-connected block by its assigned tree. Note that this yields a weakly addable class $\mathcal{C}_n$ such that all of its elements are forests. Moreover, there is a bijection between $\mathcal{E}_n$ and $\mathcal{C}_n$ such that each graph $G \in \mathcal{E}_n$ has the same number of components as its image. Hence if at least an x fraction of the graphs in $\mathcal{C}_n$ is connected, then the same is true for the equivalence class $\mathcal{E}_n$ and similarly for all other equivalence classes. This in turn implies the result. $\square$

Now, we can prove Theorem 1.2.

Proof. Because of Lemma 2.1, it remains to prove Theorem 1.2 if $\mathcal{A}$ consists of forests only. Let $\mathcal{A}_n^i \subseteq \mathcal{A}_n$ be the set of forests of $\mathcal{A}$ on $\{1,\ldots,n\}$ with i components. Thus $\mathcal{A}_n^1$ consists of trees on n vertices. Assume that there exists an $0 \leq x \leq 1$ such that

$$i|\mathcal{A}_n^{i+1}| \leq x|\mathcal{A}_n^i| \qquad \text{for all } i = 1,\ldots,\lfloor \log n \rfloor, \tag{1}$$

and

$$i|\mathcal{A}_n^{i+1}| \leq |\mathcal{A}_n^i| \qquad \text{for all } i = \lfloor \log n \rfloor + 1,\ldots,n-1. \tag{2}$$

Then either $|\mathcal{A}_n^i| = 0$ or

$$|\mathcal{A}_n^i| = \frac{|\mathcal{A}_n^i|}{|\mathcal{A}_n^{i-1}|}\frac{|\mathcal{A}_n^{i-1}|}{|\mathcal{A}_n^{i-2}|}\cdots\frac{|\mathcal{A}_n^2|}{|\mathcal{A}_n^1|}|\mathcal{A}_n^1|$$

and hence for n sufficiently large

$$\frac{\sum_{i=1}^{n}|\mathcal{A}_n^i|}{|\mathcal{A}_n^1|} \le \sum_{i=1}^{\lfloor \log n\rfloor} \frac{x^{i-1}}{(i-1)!} + \sum_{i=\lfloor \log n\rfloor+1}^{n} \frac{x^{\lfloor \log n\rfloor -1}}{(i-1)!} \le \sum_{i=0}^{\infty}\frac{x^i}{i!} + \sum_{i=\lfloor \log n\rfloor}^{\infty}\frac{1}{i!} = e^x + o(1).$$

Thus

$$\mathbb{P}[R_n \text{ is connected}] = \frac{|\mathcal{A}_n^1|}{\sum_{i=1}^{n}|\mathcal{A}_n^i|} \ge e^{-x} + o(1).$$

To prove (2), we consider the bipartite graph $B = (\mathcal{A}_n^i \cup \mathcal{A}_n^{i+1}, E)$ with an edge in E between a forest $F \in \mathcal{A}_n^i$ and a forest $F' \in \mathcal{A}_n^{i+1}$ if F' can be obtained from F by removing an edge. Since any forest in $\mathcal{A}_n^i$ has $n-i$ edges, such a forest is adjacent to at most $n-i$ forests in $\mathcal{A}_n^{i+1}$. In addition, as the class $\mathcal{A}$ is weakly addable, each forest in $F' \in \mathcal{A}_n^{i+1}$ with components of size $k_1, \ldots, k_{i+1}$ is adjacent to $\sum_{j=1}^{i+1}\sum_{l=j+1}^{i+1} k_j k_l$ forests in $\mathcal{A}_n^i$. As $k_j \ge 1$, it follows that

$$\begin{aligned}\sum_{j=1}^{i+1}\sum_{l=j+1}^{i+1} k_j k_l &= \sum_{j=1}^{i+1}\sum_{l=j+1}^{i+1}((k_j-1)(k_l-1)+k_j+k_l-1) \ge \sum_{j=1}^{i+1}\sum_{l=j+1}^{i+1}(k_j+k_l-1)\\ &= i\sum_{j=1}^{i+1}k_j - \frac{i(i+1)}{2} = i(n-i) + \binom{i}{2} \ge i(n-i).\end{aligned}$$

thus each forest in $\mathcal{A}_n^{i+1}$ is adjacent to at least $i(n-i)$ forests in $\mathcal{A}_n^i$ (and in fact any forest consisting of i isolated vertices and one component of size $n-i$ has minimal degree in B). Counting the edges of B in two different ways yields

$$|\mathcal{A}_n^i|(n-i) \ge |\mathcal{A}_n^{i+1}| i(n-i)$$

and (2) follows.

To prove (1) we again consider the graph B but with a weighting on the edges. We assign to the edge $\{F, F'\}$ a weight depending on the degrees of the endvertices of the edge $\{u, v\}$ that we remove from $F \in \mathcal{A}_n^i$ to obtain $F' \in \mathcal{A}_n^{i+1}$. More precisely, for some fixed α, $0 < \alpha \le 1$, that we shall determine later, we assign the weight $1/(d(u)d(v))^\alpha$ to $\{F, F'\}$ where $d(u), d(v)$ are the degrees of u and v in F.

Consider a forest F' consisting of trees $T_1, \ldots, T_{i+1}$. Because the class $\mathcal{A}_n$ is weakly addable, every forest that is obtained by adding an edge between two trees T_i, T_j is in $\mathcal{A}_n^i$. Hence the sum of the weight over all edges incident to F' equals

$$\sum_{\substack{i<j\\ u\in T_i, v\in T_j}} \frac{1}{(d(u)+1)^\alpha}\frac{1}{(d(v)+1)^\alpha} = \sum_{i<j}\left(\sum_{u\in T_i}\frac{1}{(d(u)+1)^\alpha}\cdot\sum_{v\in T_j}\frac{1}{(d(v)+1)^\alpha}\right).$$

We want to give a lower bound on this sum of weights. To do so we consider

$$\min_{T,|V(T)|=n} \sum_{u\in T} \frac{1}{(d(u)+1)^{\alpha}}$$

where the minimum is taken over all trees with n vertices. Note that $\sum_{u\in T} d(u) = 2n-2$. In addition $(x+1)^{-\alpha}$ is a convex function and hence it is minimised if $d(u) \le d(v)+1$ for all $u, v \in T$. Thus the minimum is attained if T is a path and

$$\min_{T,|V(T)|=n} \sum_{u\in T} \frac{1}{(d(u)+1)^{\alpha}} = \frac{n-2}{3^{\alpha}} + \frac{2}{2^{\alpha}} \ge \frac{n}{3^{\alpha}}.$$

If the forest $F' \in \mathcal{A}_n^{i+1}$ consists of components of size $k_1, \dots, k_{i+1}$, then the sum of the weights over all edges incident to F' in B is at least $3^{-\alpha} \sum_{j=1}^{i+1} \sum_{l=j+1}^{i+1} k_j k_l$ which is at least $3^{-\alpha} i(n-i)$ as before.

To obtain an upper bound on the sum of weights over all edges incident to a forest $F \in \mathcal{A}_n^i$, we consider a tree T and $R_{-\alpha}(T) = \sum_{\{v,u\}\in E(T)} (d(u)d(v))^{-\alpha}$. The value $R_{-\alpha}(T)$ is called the Randić index. It is known [1] that there exists a computable constant $\beta_0(\alpha)$ such that for each tree on at least 3 vertices $R_{-\alpha}(T) \le \beta_0(\alpha)(n+1)$. Hence in general, $R_{-\alpha}(T) \le \beta_0(\alpha)(n+C)$ for some constant C. It follows that for all $F \in \mathcal{A}_n^i$ the sum of weights over all edges incident to F is at most $\beta_0(\alpha)(n+Ci)$.

Thus by counting the edge weights of the bipartite graph in two different ways we obtain

$$|\mathcal{A}_n^i|\ \beta_0(\alpha)(n+Ci) \ge |\mathcal{A}_n^{i+1}| \frac{i(n-i)}{3^{\alpha}},$$

and thus

$$\frac{|\mathcal{A}_n^{i+1}|}{|\mathcal{A}_n^i|} \le \frac{\beta_0(\alpha)(n+Ci)}{i(n-i)/3^{\alpha}} \le \frac{\beta_0(\alpha)3^{\alpha}}{i}\left(1+O\left(\frac{i}{n-i}\right)\right).$$

As $i \le \log n$, it remains to find α such that $\beta_0(\alpha)3^{\alpha}$ is as small as possible. Using the algorithm described in [1] we computed $\beta_0(\alpha)$ for various values of α to estimate the optimal value of α. Setting $\alpha = 0.868$ yields $\beta_0(\alpha) \le 0.30762$ and $3^{\alpha}\beta_0(\alpha) < 0.7983$, and the claimed result follows. □

3 2-addable graph classes

To prove that a random element of a 2-addable graph class is asymptotically almost surely connected, we need the following results on the number of proper 2-edge cuts of a connected graph. We call a cut $e_1, \dots, e_\ell$ *proper* if removing any proper subset of $e_1, \dots, e_\ell$ yields a connected graph. Thus a proper 1-cut is a bridge, and a proper 2-cut e, e' is a 2-cut such that neither e nor e' is a bridge.

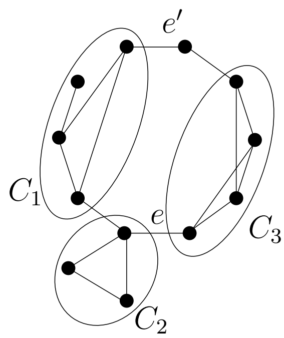

Figure 2: The components C_i as in the proof of Lemma 3.1

Lemma 3.1. *Let $G = (V, E)$ be a connected graph on n vertices and let $w : V \to \mathbb{R}^+$ be a weighting of the vertices. Then there are at most n proper edge cuts of size at most 2 that partition the vertex set into two sets V', V'' such that $\sum_{v \in V'} w(v) = r$. Moreover, the only graph on n vertices with n such proper edge cuts is the cycle C_n and in this case the weights must be periodic and of value at most r.*

Proof. Define an r-good cut to be a proper edge cut of size at most 2 that partitions the vertex set into two sets V', V'' such that $\sum_{v \in V'} w(v) = r$. Note that each r-good cut is either a bridge or a proper 2-edge cut.

We prove the result by induction on n. For $n = 2$, the only connected graph is the graph with one edge. Thus any connected graph on two vertices has at most $1 = n - 1$ edge cut and thus at most $n - 1$ r-good cuts.

Now assume that we have shown the result for all graphs on $1, \ldots, n-1$ vertices and all weightings $w : V \to \mathbb{R}^+$. Consider a graph $G = (V, E)$ on n vertices and a weighting $w : V \to \mathbb{R}^+$. Let $e = \{u, v\}, e'$ be an r-good 2-edge cut. (If none exists then all r-good cuts are bridges and hence there are at most $n - 1$ of them.) Removing e from G yields a (1-edge-)connected graph G'. Now, consider a shortest path P connecting u and v in G'. Such a path exists since e, e' is a proper 2-edge cut. The path P consists of some edges that are bridges in G' (for example e' is such an edge). Removing these bridges may yield non-trivial components in G', say $C_1, \ldots, C_k$; see Figure 2. Observe, that every proper 2-edge-cut of G is either contained in some C_i, $i = 1, \ldots, k$, or contained in the set of bridges of G' and e.

If $k = 0$ then G is a cycle. A cycle on n vertices has at most n r-good cuts, as any edge $e = \{a, b\}$ in the cycle can belong to at most two r-good cuts and each r-good cut consists of two edges. To see this, note that each cut e, e' separates the cycle into two paths at least one of which has to have total weight r. This path with total weight r has to avoid e but has either a or b as an endpoint. Since all the weights are strictly positive it follows that there can be at most one path starting at a avoiding e with total weight r, and at most one path starting at b avoiding e with total weight r.

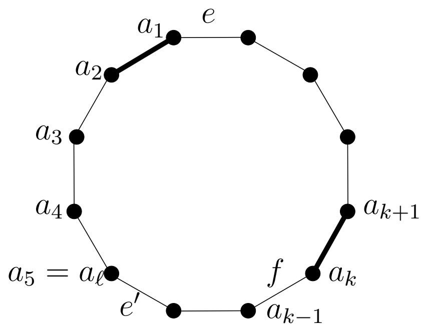

Figure 3: A configuration that is not possible if there are n r-good cuts

If $k = 0$ and G has exactly n r-good cuts, then every edge must belong to exactly two r-good 2-edge cuts. Consider the r-good cut e, e' that partitions the vertex set of G into a path $a_1, \ldots, a_\ell$ with $\sum_{i=1}^{\ell} w(a_i) = r$ and a path $a_{\ell+1}, \ldots, a_n$. Let a_1, a_n be incident to e and $a_\ell, a_{\ell+1}$ be incident to e'. We claim that $\{a_1, a_2\}$ has to form an r-good cut with the edge $\{a_{\ell+1}, a_{\ell+2}\}$ and that $\sum_{i=2}^{\ell+1} w(a_i) = r$. It then follows that $w(a_1)$ equals $w(a_{\ell+1})$ and hence the weighting is periodic. To prove the claim consider the edge $\{a_1, a_2\}$. As this edge has to be in two r-good cuts there exists a $k \geq 2$ such that $\{a_1, a_2\}$ forms an r-good cut with $\{a_k, a_{k+1}\}$ and $\sum_{i=2}^{k} w(a_i) = r$. As $r = \sum_{i=1}^{\ell} w(a_i) > \sum_{i=2}^{\ell} w(a_i)$, we have $k > \ell$. Assume $k > \ell + 1$ and consider the edge $f = \{a_{k-1}, a_k\}$; see Figure 3. As f has to belong to two r-good cuts, there must be a an r-good cut f, f' such that a_{k-1} belongs to a path $P' = a_p, \ldots, a_{k-2}, a_{k-1}$ with

$$\sum_{i=p}^{k-1} w(a_i) = r. \tag{3}$$

As $r = \sum_{i=1}^{\ell} w(a_i) < \sum_{i=1}^{k-1} w(a_i)$, we have $p \geq 2$. But then $r = \sum_{i=2}^{k} w(a_i) > \sum_{i=p}^{k-1} w(a_i)$ contradicting (3). Hence $k = \ell + 1$ and the claim is proved.

If $k \geq 1$ let $b_1, \ldots, b_j$ be the maximal subpath of P contained in C_1. If $j = 1$ (that is b_1 is a cut-vertex), then set C_1' equal to $G \setminus C_1 \cup \{b_1\}$ so that G is obtained by merging C_1 and C_1' at the cut-vertex b_1. We contract C_1 to b_1 and set $w(b_1) = \sum_{v \in C_1} w(v)$ to obtain G', and contract C_1' to b_1 and set $w(b_1) = \sum_{v \in C_1'} w(v)$ to obtain G''. If $\sum_{v \in C_1} w(v) \leq r$ then there are no r-good cuts in C_1 and it follows by induction that G' and thus G has at most $|V(G')| \leq n - 1$ good r-cuts. Similarly if $\sum_{v \in C_1'} w(v) \leq r$ then G has at most $n - 1$ good r-cuts. Now, if $\sum_{v \in C_1} w(v) > r$ and $\sum_{v \in C_1} w(v) > r$ then neither G' nor G'' is a cycle with (periodic) edge weights at most r and thus by the induction hypothesis the number of r-good cut is at most $|V(G')| - 1 + |V(G'')| - 1 = n - 1$.

If $j \geq 2$, let G_1 be the graph obtained from C_1 by identifying b_1 and b_j to form a new vertex a with weight $b_1 + b_j + \sum_{v \in V \setminus V(C_1)} w(v)$, and let G_2 be the graph obtained from G by identifying all vertices of C_1 to a single vertex b with weight $\sum_{v \in C_1} w(v)$. Note that in any proper 2-edge cut of G contained in C_1, the vertices b_1 and b_j are in the same partition class, and thus any r-good cut in G is either a cut in G_1 or in G_2. Furthermore an r-good 2-edge cut in G contained in C_1 is either an r-good 2-edge cut in G_1 or a bridge (containing a) and thus remains an r-good cut. Also if $|V(G_2)| = 2$ then G_2 consists of an edge representing (the r-good cut) e, e' and is an r-good cut itself. If $\sum_{v\in C_1} w(v) \le r$ then there are no r-good cuts in C_1 and it follows by induction that G_2 and thus G has at most $|V(G_2)| \le n-1$ good r-cuts. If $\sum_{v\in C_1} w(v) > r$ then G_2 is not a cycle with (periodic) edge weights at most r and thus by the induction hypothesis the number of r-good cut in G_2 is at most $|V(G_2)|-1$. Also by our induction hypothesis the number of r-good cuts in G_1 is at most $|V(G_1)|$ and thus the number of r-good cuts in G is at most $|V(G_1)|-1+|V(G_2)| = |V(G)|-1 = n-1$. $\square$

Now we can prove Theorem 1.3

Proof. Let $\mathcal{B}_n^i \subseteq \mathcal{B}_n \neq \emptyset$ be the set of graphs of $\mathcal{B}$ on $\{1,\dots,n\}$ with i components. Thus $\mathcal{B}_n^1$ consists of all connected graphs of $\mathcal{B}_n$. It follows from (1) and (2) that $i|\mathcal{B}_n^{i+1}| \le |\mathcal{B}_n^i|$ for all $i = 1,\dots,n$. For $i=1$ we shall see that something much stronger is true, namely $n|\mathcal{B}_n^2| \le 14|\mathcal{B}_n^1|$. It then follows that for $i \ge 2$ and $|\mathcal{B}_n^i| \neq 0$,

$$|\mathcal{B}_n^i| = \frac{|\mathcal{B}_n^i|}{|\mathcal{B}_n^{i-1}|}\frac{|\mathcal{B}_n^{i-1}|}{|\mathcal{B}_n^{i-2}|}\cdots\frac{|\mathcal{B}_n^2|}{|\mathcal{B}_n^1|}|\mathcal{B}_n^1| \le \frac{14}{n(i-1)!}|\mathcal{B}_n^1|.$$

Thus

$$\frac{\sum_{i=1}^n |\mathcal{B}_n^i|}{|\mathcal{B}_n^1|} \le 1 + \frac{14}{n}\sum_{i=2}^n \frac{1}{(i-1)!} \le 1 + \frac{14e}{n},$$

and hence

$$\mathbb{P}[R_n' \text{ is connected}] = \frac{|\mathcal{B}_n^1|}{\sum_{i=1}^n |\mathcal{B}_n^i|} \ge 1 - o(1).$$

To prove that $n|\mathcal{B}_n^2| \le 14|\mathcal{B}_n^1|$ consider the bipartite graph $B = (\mathcal{B}_n^1 \cup \mathcal{B}_n^2, E)$ with an edge between a graph $G \in \mathcal{B}_n^1$ and a graph $G' \in \mathcal{B}_n^2$ if G can be obtained from $\mathcal{B}_n^2$ by adding at most two edges between the components of $\mathcal{B}_n^2$. Equivalently there is an edge between G and G' if G' can be obtained by removing the edges of a proper edge-cut of size at most 2 from G. We partition $\mathcal{B}_n^2$ into $\lfloor n/2 \rfloor$ classes $\mathcal{C}_1,\dots,\mathcal{C}_{\lfloor n/2\rfloor}$ in such a way that $\mathcal{C}_i$ consists of graphs with one component of size i and the other of size $n-i$. Note that in the bipartite graph B, each graph in $\mathcal{C}_i$ is adjacent to $\binom{i(n-i)+1}{2}$ graphs in $\mathcal{B}_n^1$ as there are $i(n-i)$ possibilities to add an edge between the components and we have to pick one or two of these. Moreover, by Lemma 3.1 with the weighting that gives each vertex the weight 1, each graph in $\mathcal{B}_n^1$ is adjacent to at most n graphs in $\mathcal{C}_i$. Thus counting the edges of B in two different ways yields

$$\binom{i(n-i)+1}{2}|\mathcal{C}_i| \le n|\mathcal{B}_n^1|.$$

Thus for large n,

$$|\mathcal{B}_n^2| = \sum_{i=1}^{\lfloor n/2 \rfloor} |\mathcal{C}_i| \le \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{2n}{i^2(n-i)^2}|\mathcal{B}_n^1| \le \frac{8|\mathcal{B}_n^1|}{n} \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{i^2} \le \frac{8\pi^2}{6n}|\mathcal{B}_n^1| \le \frac{14}{n}|\mathcal{B}_n^1|.$$

□

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