Connectivity of random addable graphs

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October 13, 2009

Abstract

A non-empty class $\mathcal{A}$ of labelled graphs is weakly addable if for each graph $G \in \mathcal{A}$ and any two distinct components of G, any graph that can be obtain by adding an edge between the two components is also in $\mathcal{A}$. For a weakly addable graph class $\mathcal{A}$, we consider a random element R_n chosen uniformly from the set of all graph in $\mathcal{A}$ on the vertex set $\{1, \ldots, n\}$. McDiarmid, Steger and Welsh conjecture [5] that the probability that R_n is connected is at least $e^{-1/2} + o(1)$ as $n \to \infty$, and showed that it is at least e^{-1} for all n. Balister, Bollobás and Gerke improved the result by showing that this probability is at least $e^{-0.7983}$ for sufficiently large n. In this paper the results on the connectivity of random addable graphs are surveyed and some extensions of the conjecture are discussed.

1 Introduction

Motivated by [5] and following [2] we call a non-empty class $\mathcal{A}$ of labelled graphs *weakly addable*, if for each graph G in $\mathcal{A}$, whenever u and v are

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vertices in distinct components of G the graph obtained from G by adding an edge joining u and v is also in $\mathcal{A}$. In [5] a weakly addable graph class is defined so that it is also closed under isomorphism, but we do not need this additional requirement. Examples of weakly addable graph classes include forests, planar graphs, and triangle-free graphs, or more generally any H-free or H-minor-free class of graphs for any 2-edge-connected graph H.

For a class $\mathcal{A}$ of labelled graphs, we let $\mathcal{A}_n$ denote the set of graphs in $\mathcal{A}$ on the vertex set $[n] = \{1, \dots, n\}$. We want to consider a random element R_n of $\mathcal{A}$, that is, we draw R_n uniformly at random from $\mathcal{A}_n$. C. McDiarmid, A. Steger and D. Welsh [5] showed several properties that hold asymptotically almost surely for R_n. (We say that a property P holds asymptotically almost surely for a random addable graph if the probability that $R_n \in P$ tends to one as n tends to infinity.) Many results in their paper relied on the fact that a constant proportion of graphs in any nonempty weakly addable graph class is connected. More precisely, C. McDiarmid, A. Steger and D. Welsh showed [4] that $\mathbb{P}[R_n \text{ is connected }] \geq 1/e$ for an element R_n drawn uniformly at random from $\mathcal{A}_n \neq \emptyset$ of an addable class $\mathcal{A}$ for any n. They conjectured that this result can be strengthened in the following way.

Conjecture 1.1. *[5] Let $\mathcal{A}$ be any weakly addable class of graphs. Suppose that $\mathcal{A}_n$ is non-empty for all sufficiently large n, and let R_n be drawn uniformly at random from $\mathcal{A}_n$. Then*

$$\liminf_{n\to\infty} \mathbb{P}[R_n \textit{ is connected}] \geq \frac{1}{\sqrt{e}}.$$

Since an element F_n chosen uniformly at random from the set $\mathcal{F}_n$ of forests with n vertices satisfies $\lim_{n\to\infty} \mathbb{P}[F_n \text{ is connected }] = 1/\sqrt{e}$ [7], the lower bound in Conjecture 1.1 cannot be strengthened. Actually, the class of forests may be in some sense the least connected class of addable graphs which is made precise in the following conjecture.

Conjecture 1.2. *Let $\mathcal{A}$ be weakly addable class of graphs, and let $\mathcal{A}_n$ be non-empty. Let R_n be drawn uniformly at random from $\mathcal{A}_n$, and let F_n be drawn uniformly at random form the class $\mathcal{F}_n$ of forests with n vertices. Then*

$$\mathbb{P}[F_n \textit{ is connected}] \leq \mathbb{P}[R_n \textit{ is connected}].$$

At the moment the best bound on the connectivity in random addable graphs is due to P. Balister, B. Bollobás and S. Gerke and is given in the following theorem.

Theorem 1.3. *[2] Let $\mathcal{A}$ be any weakly addable class of graphs. Suppose that $\mathcal{A}_n$ is non-empty, and let R_n be drawn uniformly at random from $\mathcal{A}_n$. Then for sufficiently large n*

$$\mathbb{P}[R_n \text{ is connected}] \geq e^{-0.7983}.$$

There have been several extension to the notion of addability. The first extension is 2-addable graphs. We say that a non-empty class $\mathcal{B}$ of labelled graphs is 2-addable, if for each graph G in $\mathcal{B}$ and for any pair (C_1, C_2) of distinct components of G, any graph obtained from G by adding at most 2 edges between C_1 and C_2 also lies in $\mathcal{B}$. Note that a 2-addable graph class is weakly addable and thus for large n the probability that an element R_n chosen uniformly at random from a 2-addable class of graph is connected is at least $e^{-0.7983}$. It is known that in fact this probability tends to 1 as n tends to infinity as stated in the next theorem.

Theorem 1.4. *[2] Let $\mathcal{B}$ be any 2-addable class of graphs. Suppose that $\mathcal{B}_n$ is non-empty for all sufficiently large n, and let R_n be drawn uniformly at random from $\mathcal{B}_n$. Then*

$$\lim_{n\to\infty} \mathbb{P}[R_n \text{ is connected}] = 1.$$

It is natural to ask about that probability that a random graph of a 2-addable class is 2-edge-connected. If asked in this way, the answer is that the probability may be zero as the class consisting of a single connected graph with a bridge is 2-addable but not 2-edge-connected. When the definition of 2-addability is slightly strengthened to 2^*-addability where for every graph G in a 2^*-addable classe $\mathcal{E}$, any graph that can be obtained by adding an edge between two vertices of G that do not have at least two internally edge-disjoint path between them in G is also in $\mathcal{E}$. For 2^*-addable graph classes it is open whether a constant fraction of graph in such a class are 2-edge-connected.

Conjecture 1.5. *There exist a constant $c > 0$ such that for every 2^*-addable class $\mathcal{E}$ with $\mathcal{E}_n \neq \emptyset$ for all sufficiently large n an element R_n drawn uniformly* *at random from $\mathcal{E}_n$ satisfies*

$$\liminf_{n\to\infty} \mathbb{P}[R_n \text{ is 2-edge-connected}] \geq c.$$

Another extension allows you to also delete bridges. Following [1] we call a class $\mathcal{D}$ downwards addable if it is weakly addable and if for each graph $G \in \mathcal{D}$ any graph G' that can be obtained from G by deleting an edge of G that increases the number of components is also in $\mathcal{D}$. All the examples given for weakly addable graphs are in fact downwards addable. In particular, forests are downwards addable and hence the best general lower bound on the probability that a random element of $\mathcal{D}_n$ is connected for a downwards addable graph class $\mathcal{D}$ cannot exceed $1/\sqrt{e}+o(1)$. It is known that this lower bound is essentially correct [6].

In the remainder of the paper we shall sketch the proof of Theorem 1.3.

2 Forests

We have already seen that the class of forests plays an important role as it is conjectured that it is the least connected graph class. The following Lemma from [2] shows that in order to prove Conjecture 1.1 it suffices to consider graph classes in which all connected components are trees.

Lemma 2.1. *[2] If for every weakly addable graph class consisting of forests only, an element R_n drawn uniformly at random from all elements of this class on $\{1,\ldots,n\}$ satisfies $\mathbb{P}[R_n$ is connected$] \geq x$ for some $0 \leq x \leq 1$, then an element R'_n drawn uniformly at random from all graphs $\{1,\ldots,n\}$ of any weakly addable graph class satisfies $\mathbb{P}[R'_n$ is connected$] \geq x$.*

Proof. Let $\mathcal{A}$ be a weakly addable graph class. We say that two graphs G, G' in $\mathcal{A}_n$ are equivalent if the graphs obtained from G and G' by removing all bridges are identical; see Figure 1. In other words, G and G' are equivalent, if they have the same 2-edge-connected blocks of size at least 3. Consider a fixed equivalence class $\mathcal{E}_n$, and the collection of 2-edge-connected blocks of size at least 3 obtained by removing all bridges from a graph $G \in \mathcal{E}_n$. Note that it does not matter which graph $G \in \mathcal{E}_n$ is chosen as all graphs in $\mathcal{E}_n$

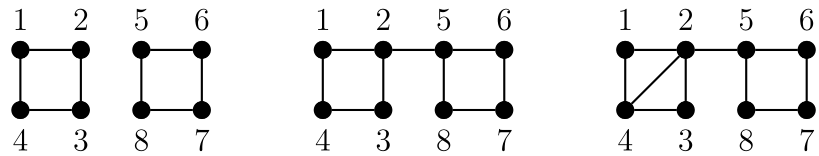

Figure 1: The first two graphs are equivalent but the third is not equivalent to the first two.

have the same 2-edge-connected blocks of size at least 3. For each such block, we fix a tree on the same set of vertices. For each graph in the equivalence class $\mathcal{E}_n$, we replace each 2-edge-connected block by its assigned tree. Note that this yields a weakly addable class $\mathcal{C}_n$ such that all of its elements are forests. Moreover, there is a bijection between $\mathcal{E}_n$ and $\mathcal{C}_n$ such that each graph $G \in \mathcal{E}_n$ has the same number of components as its image. Hence if at least an x fraction of the graphs in $\mathcal{C}_n$ is connected, then the same is true for the equivalence class $\mathcal{E}_n$ and similarly for all other equivalence classes. This in turn implies the result. $\square$

Note that if we are interested in downwards addable graphs Lemma 2.1 implies that one only has to consider classes of addable graphs whose minimal elements are graphs consisting of a collection of say cliques. (Here, we mean that an element is minimal if it is minimal in the natural partial order on graphs where a graph H is smaller than G when H is a subgraph of G.)

3 Proof of Theorem 1.3

Because of Lemma 2.1, it remains to prove Theorem 1.3 if $\mathcal{A}$ consists of forests only. Let $\mathcal{A}_n^i \subseteq \mathcal{A}_n$ be the set of forests of $\mathcal{A}$ on $\{1, \ldots, n\}$ with i components. Thus $\mathcal{A}_n^1$ consists of trees on n vertices. Assume that there exists an $0 \leq x \leq 1$ such that

$$i|\mathcal{A}_n^{i+1}| \leq x|\mathcal{A}_n^i| \qquad \text{for all } i = 1, \ldots, \lfloor \log n \rfloor, \tag{1}$$

and

$$i|\mathcal{A}_n^{i+1}| \leq |\mathcal{A}_n^i| \qquad \text{for all } i = \lfloor \log n \rfloor + 1, \ldots, n-1. \tag{2}$$

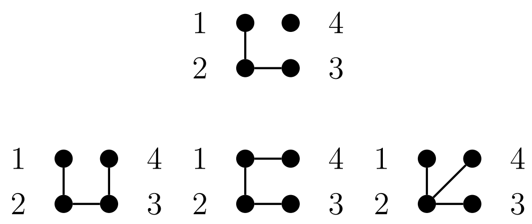

Figure 2: The graph on the top is adjacent to all the graphs on the bottom

Then either $|\mathcal{A}_n^i| = 0$ or

$$|\mathcal{A}_n^i| = \frac{|\mathcal{A}_n^i|}{|\mathcal{A}_n^{i-1}|}\frac{|\mathcal{A}_n^{i-1}|}{|\mathcal{A}_n^{i-2}|}\cdots\frac{|\mathcal{A}_n^2|}{|\mathcal{A}_n^1|}|\mathcal{A}_n^1|$$

and hence for n sufficiently large

$$\frac{\sum_{i=1}^n |\mathcal{A}_n^i|}{|\mathcal{A}_n^1|} \leq \sum_{i=1}^{\lfloor \log n \rfloor} \frac{x^{i-1}}{(i-1)!} + \sum_{i=\lfloor \log n \rfloor+1}^{n} \frac{x^{\lfloor \log n \rfloor -1}}{(i-1)!} \leq \sum_{i=0}^{\infty} \frac{x^i}{i!} + \sum_{i=\lfloor \log n \rfloor}^{\infty} \frac{1}{i!} = e^x + o(1).$$

Thus

$$\mathbb{P}[R_n \text{ is connected}] = \frac{|\mathcal{A}_n^1|}{\sum_{i=1}^n |\mathcal{A}_n^i|} \geq e^{-x} + o(1).$$

To prove (2), we consider the bipartite graph $B = (\mathcal{A}_n^i \cup \mathcal{A}_n^{i+1}, E)$ with an edge in E between a forest $F \in \mathcal{A}_n^i$ and a forest $F' \in \mathcal{A}_n^{i+1}$ if F' can be obtained from F by removing an edge, see Figure 2. Since any forest in $\mathcal{A}_n^i$ has $n-i$ edges, such a forest is adjacent to at most $n-i$ forests in $\mathcal{A}_n^{i+1}$. In addition, as the class $\mathcal{A}$ is weakly addable, each forest in $F' \in \mathcal{A}_n^{i+1}$ with components of size $k_1, \ldots, k_{i+1}$ is adjacent to $\sum_{j=1}^{i+1}\sum_{l=j+1}^{i+1} k_j k_l$ forests in $\mathcal{A}_n^i$. In particular, not all vertices in our bipartite graph have the same degree, see Figure 2 and Figure 3.

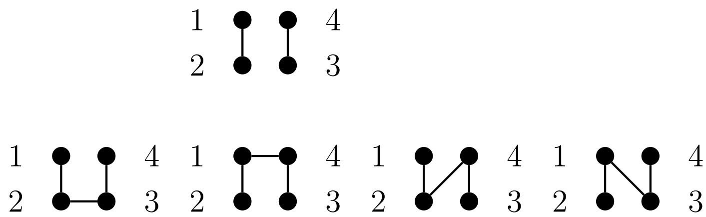

Figure 3: The graph on top is adjacent to all graphs on the bottom

As $k_j \geq 1$, it follows that

$$\begin{aligned}\sum_{j=1}^{i+1}\sum_{l=j+1}^{i+1} k_j k_l &= \sum_{j=1}^{i+1}\sum_{l=j+1}^{i+1}((k_j-1)(k_l-1)+k_j+k_l-1) \geq \sum_{j=1}^{i+1}\sum_{l=j+1}^{i+1}(k_j+k_l-1)\\ &= i\sum_{j=1}^{i+1}k_j - \frac{i(i+1)}{2} = i(n-i)+\binom{i}{2} \geq i(n-i).\end{aligned}$$

thus each forest in $\mathcal{A}_n^{i+1}$ is adjacent to at least $i(n-i)$ forests in $\mathcal{A}_n^i$ (and in fact any forest consisting of i isolated vertices and one component of size $n-i$ has minimal degree in B). Counting the edges of B in two different ways yields

$$|\mathcal{A}_n^i|(n-i) \geq |\mathcal{A}_n^{i+1}|i(n-i)$$

and (2) follows.

To prove (1) we again consider the graph B but with a weighting on the edges. Ideally, we would like to find a weighting such that the weighted degree of each vertex is the same. Such an ideal weighting would yield the optimal result immediately. However, it is conceivable that such a weighting does not exist even though the conjecture is true which gives a further indication why the proof of the conjecture does not seem to be straightforward.

We use the following weighting to obtain our bound. We assign to the edge $\{F, F'\}$ a weight depending on the degrees of the endvertices of the edge

$\{u,v\}$ that we remove from $F \in \mathcal{A}_n^i$ to obtain $F' \in \mathcal{A}_n^{i+1}$. More precisely, for some fixed α, $0 < \alpha \leq 1$, that we shall determine later, we assign the weight $1/(d(u)d(v))^\alpha$ to $\{F, F'\}$ where $d(u), d(v)$ are the degrees of u and v in F.

Consider a forest F' consisting of trees $T_1, \dots, T_{i+1}$. Because the class $\mathcal{A}_n$ is weakly addable, every forest that is obtained by adding an edge between two trees T_i, T_j is in $\mathcal{A}_n^i$. Hence the sum of the weight over all edges incident to F' equals

$$\sum_{\substack{i<j \\ u \in T_i, v \in T_j}} \frac{1}{(d(u)+1)^\alpha} \frac{1}{(d(v)+1)^\alpha} = \sum_{i<j} \left(\sum_{u \in T_i} \frac{1}{(d(u)+1)^\alpha} \cdot \sum_{v \in T_j} \frac{1}{(d(v)+1)^\alpha} \right). \tag{3}$$

We want to give a lower bound on this sum of weights. To do so we consider

$$\min_{T, |V(T)|=n} \sum_{u \in T} \frac{1}{(d(u)+1)^\alpha}$$

where the minimum is taken over all trees with n vertices. Note that $\sum_{u \in T} d(u) = 2n-2$. In addition $(x+1)^{-\alpha}$ is a convex function and hence it is minimised if $d(u) \leq d(v)+1$ for all $u, v \in T$. Thus the minimum is attained if T is a path and

$$\min_{T, |V(T)|=n} \sum_{u \in T} \frac{1}{(d(u)+1)^\alpha} = \frac{n-2}{3^\alpha} + \frac{2}{2^\alpha}.$$

Thus if the forest $F' \in \mathcal{A}_n^{i+1}$ consists of components of size $k_1 \geq \ldots \geq k_{i+1}$, then the graph consisting of paths $P_1, \dots, P_{i+1}$ of length $k_1, \dots, k_{i+1}$ has smaller weighted degree in our bipartite graph. Moreover if $k_2 \geq 3$ then the graph consisting of paths $P_1', \dots, P_{i+1}'$ of length $k_1+1, k_2-1, k_3, \dots, k_{i+1}$ has a weighted degree as most as large as one can see as follows. First

$$\begin{aligned} \left(\frac{k_1-2}{3^\alpha} + \frac{2}{2^\alpha}\right)\left(\frac{k_2-2}{3^\alpha} + \frac{2}{2^\alpha}\right) &= \frac{k_1k_2 - 2k_1 - 2k_2 + 4}{3^{2\alpha}} + \frac{2k_1 + 2k_2 - 8}{2^\alpha 3^\alpha} + \frac{4}{2^{2\alpha}} \\ &\geq \frac{k_1k_2 - 3k_1 - k_2 + 3}{3^{2\alpha}} + \frac{2k_1 + 2k_2 - 8}{2^\alpha 3^\alpha} + \frac{4}{2^{2\alpha}} \\ &= \left(\frac{k_1-1}{3^\alpha} + \frac{2}{2^\alpha}\right)\left(\frac{k_2-3}{3^\alpha} + \frac{2}{2^\alpha}\right) \end{aligned}$$

and thus the summand in (3) with $i=1$, $j=2$ is at least as large when considering P_1 and P_2 instead of P_1' and P_2'. All summands that do not contain P_1, P_2 and P_1', P_2' respectively, do not change, and finally for all $j = 3, \ldots, i+1$, we have

$$\left(\sum_{u\in P_1} \frac{1}{(d(u)+1)^\alpha} \cdot \sum_{v\in P_j} \frac{1}{(d(v)+1)^\alpha}\right) + \left(\sum_{u\in P_2} \frac{1}{(d(u)+1)^\alpha} \cdot \sum_{v\in P_j} \frac{1}{(d(v)+1)^\alpha}\right)$$
$$= \left(\frac{(k_1+2)+(k_1+2)}{3^\alpha} + \frac{4}{2^\alpha}\right) \sum_{v\in P_j} \frac{1}{(d(v)+1)^\alpha}$$
$$= \left(\frac{(k_1+3)+(k_1+1)}{3^\alpha} + \frac{4}{2^\alpha}\right) \sum_{v\in P_j} \frac{1}{(d(v)+1)^\alpha}$$
$$= \left(\sum_{u\in P_1'} \frac{1}{(d(u)+1)^\alpha} \cdot \sum_{v\in T_j} \frac{1}{(d(v)+1)^\alpha}\right) + \left(\sum_{u\in P_2'} \frac{1}{(d(u)+1)^\alpha} \cdot \sum_{v\in T_j} \frac{1}{(d(v)+1)^\alpha}\right).$$

Thus the weighted degree of a graph in $\mathcal{A}_{i+1}$ attains it minimum at a graphs that contains one long path of size at least $n-2i$ and the remaining i components consist of one or two vertices. Note that an isolated vertex v satisfies $1/(d(v)+1)^\alpha = 1$ and a component with two vertices u, v satisfies $1/(d(u)+1)^\alpha + 1/(d(v)+1)^\alpha = 2/2^\alpha \geq 1$. By only considering the summands involving the large component of such a minimum weight graph one obtains for any forest F' consisting of trees $T_1, \ldots, T_{i+1}$, that the sum of the weight over all edges incident to F' satisfies

$$\sum_{\substack{k<j \\ u\in T_k,\, v\in T_j}} \frac{1}{(d(u)+1)^\alpha} \frac{1}{(d(v)+1)^\alpha} = \sum_{k<j} \left(\sum_{u\in T_k} \frac{1}{(d(u)+1)^\alpha} \cdot \sum_{v\in T_j} \frac{1}{(d(v)+1)^\alpha}\right)$$
$$\geq \sum_{j=2}^{i+1} \left(\frac{n-2i-2}{3^\alpha} + \frac{2}{2^\alpha}\right) \geq \frac{i(n-2i)}{3^\alpha}. \tag{4}$$

To obtain an upper bound on the sum of weights over all edges incident to a forest $F \in \mathcal{A}_n^i$, we consider a tree T and $R_{-\alpha}(T) = \sum_{\{v,u\}\in E(T)} (d(u)d(v))^{-\alpha}$. The value $R_{-\alpha}(T)$ is called the Randić index. It is known [3] that there exists a computable constant $\beta_0(\alpha)$ such that for each tree on at least 3 vertices $R_{-\alpha}(T) \leq \beta_0(\alpha)(n+1)$, see Figure 4. Hence in general, $R_{-\alpha}(T) \leq$

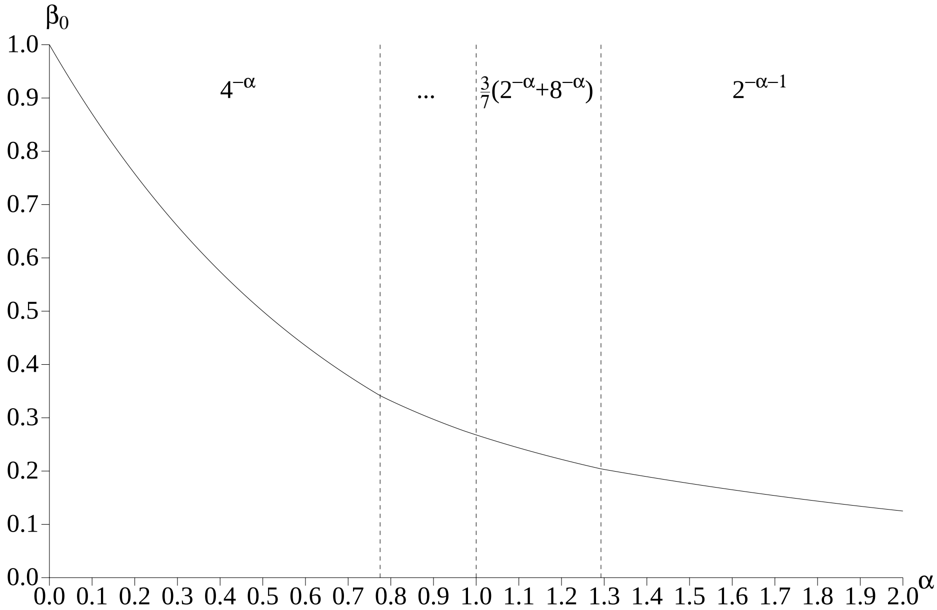

Figure 4: The function $\beta_c(\alpha)$

$\beta_0(\alpha)(n+C)$ for some constant C. It follows that for all $F \in \mathcal{A}_n^i$ the sum of weights over all edges incident to F is at most $\beta_0(\alpha)(n+Ci)$.

Thus by counting the edge weights of the bipartite graph in two different ways by (4) we obtain

$$|\mathcal{A}_n^i|\ \beta_0(\alpha)(n+Ci) \geq |\mathcal{A}_n^{i+1}| \frac{i(n-2i)}{3^\alpha},$$

and thus

$$\frac{|\mathcal{A}_n^{i+1}|}{|\mathcal{A}_n^i|} \leq \frac{\beta_0(\alpha)(n+Ci)}{i(n-2i)/3^\alpha} \leq \frac{\beta_0(\alpha)3^\alpha}{i}\left(1+O\left(\frac{i}{n-i}\right)\right).$$

As $i \leq \log n$, it remains to find α such that $\beta_0(\alpha)3^\alpha$ is as small as possible. Using the algorithm described in [3] we computed $\beta_0(\alpha)$ for various values of α to estimate the optimal value of α. Setting $\alpha = 0.868$ yields $\beta_0(\alpha) \leq 0.30762$ and $3^\alpha\beta_0(\alpha) < 0.7983$, and the claimed result follows.

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