Random partial orders defined by angular domains

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Abstract

The d-dimensional random partial order is the intersection of d independently and uniformly chosen (with replacement) linear orders on the set $[n] = \{1, 2, \ldots, n\}$. This is equivalent to picking n points uniformly at random in the d-dimensional unit cube $Q_d = [0,1]^d$ with the coordinate-wise ordering. If $d = 2$, then this can be rephrased by declaring that for any pair $P_1, P_2 \in Q_2$ we have $P_1 \prec P_2$ if and only if P_2 lies in the positive upper quadrant defined by the two axis-parallel lines crossing at P_1. In this paper we study the random partial order with parameter α ($0 \le \alpha \le \pi$) which is generated by picking n points uniformly at random from Q_2 equipped with the same partial order as above but with the quadrant replaced by an angular domain of angle α.

1   Introduction

There exists several models that generate random partial orders on n elements. A very good, but not very recent survey is given in [9]. One of the most studied models (introduced by Winkler [16, 17]) is to consider the intersection of d randomly and independently chosen linear orders on the same n element ground set. This is called the *d-dimensional random partial order*. One of the nice features of this model is that it has the same distribution as n uniformly chosen points in the d-dimensional unit cube $Q_d = [0,1]^d$ with respect to coordinate-wise ordering. Sometimes it is convenient to consider the model in which points of Q_d are generated according to a Poisson process of intensity n. Standard techniques (see e.g., [3, 4, 9]) show that the 'same' results hold in both models for all problems studied so far, and therefore

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for convenience we shall often prove results for one model and use it to deduce the corresponding result for the other. The problem that attracted the most attention of researchers is finding the largest *chain* or *antichain* (respectively a set of pairwise comparable or incomparable elements). This problem was already raised by Ulam [14] in the case $d = 2$ where the question is equivalent to that of finding the longest increasing subsequence of a random permutation. The right order of magnitude was determined by Hammersley [11], and the asymptotically precise answer was given by Logan and Schepp [13] and Vershik and Kerov [15]. Their result states that the size L_n of the largest chain is concentrated around its mean, which is $(2 + o(1))\sqrt{n}$. Research on how sharp the concentration was initialized by Frieze [10], Bollobás and Brightwell [3] and Bollobás and Johansson [5]. An upper bound on the variance of L_n that later turned out to be tight was established by Kim [12] and Baik, Deift and Johansson proved the matching lower bound, moreover they determined the limiting distribution of $\frac{L_n - 2\sqrt{n}}{n^{1/6}}$. A survey on the method of their proof (together with some applications) is given by Aldous and Diaconis in [1].

The problem was partially solved for general d by Bollobás and Winkler [7].

In this paper, we consider a generalization of the $d = 2$ case. Note that in the coordinate-wise ordering for any 2 points P_1, P_2 in the plane we have $P_1 \prec P_2$ if and only if P_2 lies in the positive upper quadrant defined by the two axis-parallel lines crossing at P_1. Modifying the angle $\frac{\pi}{2}$ to any angle α, $0 \leq \alpha \leq \pi$, we obtain a partial order that we shall call the α-ordering. More precisely, for any pair of points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbb{R}^2$ we define $P_1 \prec_\alpha P_2$ if and only if $x_1 + y_1 < x_2 + y_2$ and P_2 lies in the open angular domain at P_1 of angle α with angle bisector parallel to the line $x = y$. The probability space of the partial order obtained by selecting n points uniformly at random from $Q := Q_2 = [0, 1]^2$ and considering the partial suborder of the α-ordering induced by these points will be denoted by $\mathcal{P}_{\alpha,n}$ and the probability space obtained by points generated by a Poisson process of intensity n in Q will be denoted by $\mathcal{P}_{\alpha,n}^P$. Note that with probability 1 both $\mathcal{P}_{\alpha,n}$ and $\mathcal{P}_{\alpha,n}^P$ form an antichain if $\alpha = 0$ and a chain (i.e., a linear or total ordering) if $\alpha = \pi$. The corresponding *comparability graphs* (in which every vertex corresponds to an element of the partial order and two vertices are joined by an edge if the corresponding elements are comparable in the partial order) will be denoted by $G_{\alpha,n}$ and $G_{\alpha,n}^P$, and we will use the notation $L_{\alpha,n}$ and $L_{\alpha,n}^P$ for the lengths of the longest chains in $\mathcal{P}_{\alpha,n}$ or $\mathcal{P}_{\alpha,n}^P$.

As we will show in Section 2, a quite natural transformation yields that $\mathcal{P}_{\alpha,n}$ is equivalent to picking n points randomly in some rhombus and consider them equipped with the original $\frac{\pi}{2}$-ordering. This will enable us to determine $L_{\alpha,n}$ for most values of α. In view of this transformation, one might question whether it is interesting enough to study these angle-parametrized models at all. Our main reason to believe so is that letting α range from 0 to π, our random partial order evolves from an antichain

to a chain and the comparability graph evolves from an empty graph to a complete graph. In this way, we can study "hitting time" problems and compare the results to other random processes (like the Erdős-Rényi graphs process [8]). More precisely, if $\mathfrak{P}$ is a *monotone* poset property (e.g. containing a chain of size at least k, having a connected comparability graph, having a comparability graph of diameter at most l, etc.), then one can define

$$\tau(\mathfrak{P}) = \lim_{\varepsilon \to 0^+} \liminf_{n \to \infty} \inf\{\alpha : \mathbb{P}(\mathcal{P}_{\alpha,n} \in \mathfrak{P}) \le 1 - \varepsilon\}.$$

We will prove that $\tau(\mathfrak{P}) = \frac{\pi}{2}$ when $\mathfrak{P}$ is the property of having a connected comparability graph.

The rest of the paper is organized as follows: Section 2 contains the above mentioned transformation and results on $L_{\alpha,n}$. In particular we prove the following theorem.

Theorem 1.1. *Let $\alpha = \alpha_n$ satisfy*

$$\omega((\log n)^2 n^{-1}) \le \alpha \le \pi - \omega(n^{-1}).$$

Then both $L_{\alpha,n}/\sqrt{n \tan \frac{\alpha}{2}}$ and $L_{\alpha,n}^P/\sqrt{n \tan \frac{\alpha}{2}}$ tend to 2 in probability as $n \to \infty$.

As a converse to Theorem 1.1 we shall also show that this result fails for both $L_{\alpha,n}$ and $L_{\alpha,n}^P$ when $\alpha = o((\log n)^2 n^{-1})$ or $\alpha = \pi - o(n^{-1})$.

Section 3 addresses the problem of determining the diameter of $G_{\alpha,n}$ when $\alpha > \frac{\pi}{2}$. Winkler [17] settled the case $\alpha = \frac{\pi}{2}$ even for general dimension d showing that the diameter is 3 with probability tending to 1 as n tends to infinity (*with high probability, w.h.p.*). Our main result is the following theorem.

Theorem 1.2. *The following holds with high probability*

$$\mathrm{diam}(G_{\alpha,n}) = \begin{cases} 1 & \text{if } \pi - \alpha = o(n^{-2}) \\ 2 & \text{if } \pi - \alpha = \omega(n^{-2}) \text{ and } \alpha - \frac{\pi}{2} = \omega(n^{-1/2}) \\ 3 & \text{if } 0 \le \alpha - \frac{\pi}{2} = o(n^{-1/2}). \end{cases}$$

As $G_{\alpha,n}^P = G_{\alpha,X}$ where X is a Poisson variable of mean n, and since $X = (1+o(1))n$ w.h.p., corresponding results also hold for $G_{\alpha,n}^P$.

In Section 3 we also study the probability of the event that $G_{\alpha,n}$ is connected for any $\alpha < \frac{\pi}{2}$. In particular, we determine the expected number of connected components of $G_{\alpha,n}$. Section 4 contains some concluding remarks and open problems.

Notation. Throughout the paper Q denotes the unit square $[0,1]^2$. Given a partial ordering on Q and a point $P \in Q$ we shall write $P^+ = \{P' \in Q : P \prec P'\}$ and $P^- = \{P' \in Q : P' \prec P\}$ for the set of elements larger, respectively smaller than P. We shall also write $P^{\|} = P^+ \cup P^- \cup \{P\}$ and $P^{\perp} = Q \setminus P^{\|}$ for the set of elements comparable, respectively incomparable to P. If $P = (x, y)$, then we will use the notation $P^+(x,y), P^-(x,y), P^{\|}(x,y), P^{\perp}(x,y)$. For a set S of elements we write $S^{\|} = \bigcap_{P \in S} P^{\|}$. For a point $P \in Q$, the lines forming $\partial P^{\perp}$ in the α-ordering $\prec_\alpha$ will be denoted by $e_{\alpha,+,P}$ and $e_{\alpha,-,P}$.

2 Longest chains, antichains

In this section, we address the problem of finding $L_{\alpha,n}$ for $\alpha \in (0, \pi)$. Note that if we denote by T the rotation by angle $\frac{\pi}{2}$ around the point $(\frac{1}{2}, \frac{1}{2})$, then for any $P_1, P_2 \in Q$ we have that P_1 and P_2 are comparable in the α-ordering if and only if TP_1 and TP_2 are incomparable in the $(\pi - \alpha)$-ordering. Thus if $W_{\alpha,n}$ denotes the size of the largest antichain in $\mathcal{P}_{\alpha,n}$, then $L_{\alpha,n}$ and $W_{\pi-\alpha,n}$ have the same distribution and hence all results about $L_{\alpha,n}$ or $L^P_{\alpha,n}$ can be converted to results about $W_{\pi-\alpha,n}$ or $W^P_{\pi-\alpha,n}$.

Let us begin with stating the theorem concerning the size of the longest chain in $\mathcal{P}_{\frac{\pi}{2},n}$ or $\mathcal{P}^P_{\frac{\pi}{2},n}$.

Theorem 2.1 (Logan, Shepp [13]; Vershik, Kerov [15]). *Both $L_{\frac{\pi}{2},n}/\sqrt{n}$ and $L^P_{\frac{\pi}{2},n}/\sqrt{n}$ tend to 2 in probability as $n \to \infty$.*

In our proofs, we will use the following concentration result:

Theorem 2.2 (Kim [12]). *If $0 < \theta \le n^{1/3}/20$, then*

$$\mathbb{P}\left(|L^P_{\frac{\pi}{2},n} - 2\sqrt{n})| > \theta n^{1/6}\right) \le \exp\left(-\frac{4\theta^{3/2}}{3} + \phi(\theta)\right),$$

where

$$\phi(\theta) := \left(\frac{\theta}{27n^{1/3}} + \frac{5\log n}{\theta^{1/2}n^{1/3}}\right)\theta^{3/2}.$$

Let R_α denote the rhombus with two of its vertices being $(0,0)$, $(1,1)$ and having angle α at these vertices. Note that the area of R_α is $\tan\frac{\alpha}{2}$. Furthermore, for any measurable subset $S \subseteq \mathbb{R}^2$, write $\mathcal{P}^P_{\alpha,n}(S)$ for the probability space of random partial orders obtained by placing points in $\mathbb{R}^2$ according to a Poisson process of intensity n and then taking the α-ordering of those points of this process that lie in S. Thus $\mathcal{P}^P_{\alpha,n} = \mathcal{P}^P_{\alpha,n}(Q) = \mathcal{P}^P_{\alpha,n}(R_{\frac{\pi}{2}})$. We shall often vary the set S, in which case we consider the Poisson process on $\mathbb{R}^2$ as being fixed and each $\mathcal{P}^P_{\alpha,n}(S)$ a partial suborder of the

same partial order $\mathcal{P}^P_{\alpha,n}(\mathbb{R}^2)$. Let T^α denote the linear transformation of the plane that fixes the line $y = x$ and takes $(1,0)$ to the vertex of R_α lying in the half-plane $x > y$ and $(0,1)$ to the vertex of R_α lying in the half-plane $x < y$. Thus the image of Q under T^α is just R_α. Clearly, for any $P_1, P_2 \in \mathbb{R}^2$ we have $P_1 \prec_{\frac{\pi}{2}} P_2$ if and only if $T^\alpha P_1 \prec_\alpha T^\alpha P_2$. Thus, we obtain the following proposition.

Proposition 2.3. The probability spaces $\mathcal{P}^P_{\frac{\pi}{2},n}$ and $\mathcal{P}^P_{\alpha,n/\tan\frac{\alpha}{2}}(R_\alpha)$ have the same distribution. $\hfill\square$

The linear transformation $(T^\alpha)^{-1}$ takes the unit square to the rhombus R_β with the property that $\tan\frac{\beta}{2} = |R_\beta| = |R_\alpha|^{-1} = (\tan\frac{\alpha}{2})^{-1}$. Thus $\beta = \pi - \alpha$. Therefore by applying $(T^\alpha)^{-1} = T^{\pi-\alpha}$ we obtain another proposition that we will use in our proofs.

Proposition 2.4. The probability spaces $\mathcal{P}^P_{\alpha,n}$ and $\mathcal{P}^P_{\frac{\pi}{2},n\tan\frac{\alpha}{2}}(R_{\pi-\alpha})$ have the same distribution. $\hfill\square$

Proof of Theorem 1.1. We first prove the result for $L^P_{\alpha,n}$ by splitting the argument into two cases.

CASE I: $\alpha \geq \frac{\pi}{2}$ and $\alpha \leq \pi - \omega(n^{-1})$.

By Proposition 2.4, $L^P_{\alpha,n}$ has the same distribution as $L^P_{\frac{\pi}{2},n\tan\frac{\alpha}{2}}(R_{\pi-\alpha})$. As $\alpha \geq \frac{\pi}{2}$ we have $R_{\pi-\alpha} \subseteq Q$, so $L^P_{\frac{\pi}{2},n\tan\frac{\alpha}{2}}(R_{\pi-\alpha}) \leq L^P_{\frac{\pi}{2},n\tan\frac{\alpha}{2}}$, which by Theorem 2.1 is at most $(2+\varepsilon)\sqrt{n\tan\frac{\alpha}{2}}$ w.h.p. as $n\tan\frac{\alpha}{2} \geq n \to \infty$. For the lower bound, let S_0 be the largest square centered at $(\frac{1}{2},\frac{1}{2})$ that lies in $R_{\pi-\alpha}$. Inductively define S_i for $i > 0$ so that S_{i+1} is the largest square in $R_{\pi-\alpha}$ centered at a point on the line $y = x$ whose bottom left corner is the top right corner of S_i. Similarly define S_{-i}, $i > 0$, as the largest square in $R_{\pi-\alpha}$ whose top right corner is the bottom left corner of S_{-i+1}. Clearly any two points $P \in S_i$, $P' \in S_j$, $i \neq j$, are comparable in the $\frac{\pi}{2}$-ordering. Thus one can obtain a chain in $\mathcal{P}^P_{\frac{\pi}{2},n\tan\frac{\alpha}{2}}(R_{\pi-\alpha})$ by taking the union of the largest chains in each $\mathcal{P}^P_{\frac{\pi}{2},n\tan\frac{\alpha}{2}}(S_i)$. Let the side length of S_i be s_i and choose ℓ minimal so that $\sum_{i=-\ell}^{\ell} s_i \geq 1 - 2\varepsilon$. Then for $|i| \leq \ell$, the point in S_i closest to $(\frac{1}{2},\frac{1}{2})$ is at least ε from the boundary of Q and so $s_i \geq \varepsilon(1 - \tan(\frac{\alpha}{2} - \frac{\pi}{4})) = \Theta(\varepsilon(\pi-\alpha))$. Thus the expected number of points in each S_i is $s_i^2 n\tan\frac{\alpha}{2} = \Omega(\varepsilon^2 n(\pi-\alpha)) = \omega(1)$ for any fixed $\varepsilon > 0$. Thus we can apply Theorem 2.1 and deduce that for n sufficiently large, each S_i contains a chain of length at least $(2-\varepsilon)s_i\sqrt{n\tan\frac{\alpha}{2}}$ with probability at least $1 - \varepsilon^2$, say. If I is the set of i such that S_i does not contain such a chain, then $\mathbb{E}(\sum_{i\in I} s_i) \leq \varepsilon^2 \sum_{i=-\ell}^{\ell} s_i \leq \varepsilon^2$. Thus with probability at least $1 - \varepsilon$, $\sum_{i\in I} s_i \leq \varepsilon$ and $\mathcal{P}^P_{\frac{\pi}{2},n\tan\frac{\alpha}{2}}(R_{\pi-\alpha})$ contains a chain of length at least

$$\sum_{i\notin I}(2-\varepsilon)s_i\sqrt{n\tan\tfrac{\alpha}{2}} \geq (2-\varepsilon)((1-2\varepsilon)-\varepsilon)\sqrt{n\tan\tfrac{\alpha}{2}} \geq (2-7\varepsilon)\sqrt{n\tan\tfrac{\alpha}{2}}$$

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as required.

CASE II: $\alpha \le \frac{\pi}{2}$ and $\alpha \ge \omega((\log n)^2/n)$.

By Proposition 2.4 we can work in $\mathcal{P}^P_{\frac{\pi}{2}, n \tan \frac{\alpha}{2}}(R_{\pi-\alpha})$. As $Q \subseteq R_{\pi-\alpha}$ we obtain $L^P_{\alpha,n} \ge (2-\varepsilon)\sqrt{n \tan \frac{\alpha}{2}}$ w.h.p. by Theorem 2.1. To prove an upper bound on $L^P_{\alpha,n}$ let us consider a pair of points P_1, P_2 in $R_{\pi-\alpha}$ generated by the Poisson process. If these points are not comparable, then they cannot be contained in a chain. Otherwise, the length L of the largest chain having these points as endpoints is distributed as $2 + L^P_{\frac{\pi}{2}, sn \tan \frac{\alpha}{2}}(Q)$, where s is the area of the axis-parallel rectangle S determined by P_1 and P_2. Indeed, S is the set of points that lie between P_1 and P_2 in the coordinate-wise ordering. Furthermore, for any axis-parallel rectangle S the distribution of $L^P_{\frac{\pi}{2}, m}(S)$ depends only on the area of S (and is independent of the ratio of the length of the sides). Observe that the largest axis-parallel rectangle in $R_{\pi-\alpha}$ is the unit square Q and thus $s \le 1$. By embedding S in a larger rectangle if necessary, we can stochastically bound each $L^P_{\frac{\pi}{2}, m}(S)$ by $L^P_{\frac{\pi}{2}, m}(Q)$. Thus by Theorem 2.2, we obtain that for a fixed pair P_1, P_2 the probability that there is a chain larger than $(2+\varepsilon)\sqrt{n \tan \frac{\alpha}{2}}$ is not more than $e^{-\frac{4}{3}\theta^{3/2} + \phi(\theta)}$ where we may take $\theta = (n \tan \frac{\alpha}{2})^{1/3}/20$. But then $\theta = \omega((\log n)^{2/3})$ by the assumption on α. Hence $e^{-\lambda^2} e^{-\frac{4}{3}\theta^{3/2} + \phi(\theta)} = o(1/n^2)$. The probability that the number of points generated by the Poisson process is greater than $2n$ (and thus the number of pairs is greater than $2n^2$) is $\exp(-\Omega(n))$, thus $L^P_{\alpha,n} \le (2+\varepsilon)\sqrt{n \tan \frac{\alpha}{2}} + 2$ w.h.p..

Finally by choosing Poisson variables X and Y of means $(1-\varepsilon)n$ and $(1+\varepsilon)n$ respectively, we have that for any $\varepsilon > 0$, $X \le n \le Y$ holds w.h.p.. Thus as $L_{\alpha,n}$ is monotonic in n, $L^P_{\alpha, (1-\varepsilon)n} = L_{\alpha,X} \le L_{\alpha,n} \le L_{\alpha,Y} = L^P_{\alpha,(1+\varepsilon)n}$ w.h.p., and so the result also follows for $L_{\alpha,n}$. $\qquad\square$

We note that if $\alpha = \pi - o(n^{-1})$ then $\sqrt{n \tan \frac{\alpha}{2}} = \omega(n)$. As there are only n Poisson points on average in Q, the conclusion of Theorem 1.1 cannot possibly hold in this case. The next theorem shows that it also fails to hold when $\alpha = o((\log n)^2/n)$.

Theorem 2.5. *Assume $\alpha = o((\log n)^2/n)$. Then*

$$\mathbb{P}\big(L^P_{\alpha,n} \ge 3\sqrt{n \tan \frac{\alpha}{2}}\big) \to 1$$

as $n \to \infty$.

Proof. By Proposition 2.4 we can work in $\mathcal{P}^P_{\frac{\pi}{2}, n \tan \frac{\alpha}{2}}(R_{\pi-\alpha})$. Let $s = \sqrt{n \tan \frac{\alpha}{2}}$ and tile $\mathbb{R}^2$ with squares $S_{i,j} = (i/4s, j/4s) + [0, 1/4s]^2$ of side length $1/4s$. For each i, let $\mathcal{C}_i = \{S_{i+j,j} : S_{i+j,j} \subseteq R_{\pi-\alpha}\}$ be the set of these squares in a diagonal sequence that lie inside $R_{\pi-\alpha}$. Then there exists a $K > 0$ so that $\mathcal{C}_i$ consists of at least $3s$ squares for all i with $|i| \le K\alpha^{-1}$. If each of the squares in $\mathcal{C}_i$ contains a point of the Poisson

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process, then one obtains a chain of length $3s$. The expected number of points in one of these squares is $1/16$, so this happens with probability $(1 - e^{-1/16})^{-3s} \geq e^{-9s}$. The events that each square in $\mathcal{C}_i$ contain a point are independent for distinct i. Thus if $K\alpha^{-1}e^{-9s} \to \infty$ then w.h.p. a chain of length $3s$ will exist. However this occurs as $s = \Theta((n\alpha)^{1/2}) = o(\log n)$ and $\alpha^{-1} = \omega(n^{1/2})$. $\qquad\square$

By comparing $L^P_{\alpha,n}$ with $L_{\alpha,n}$ we deduce that if $\alpha = o((\log n)^2/n)$ then $\mathbb{P}(L_{\alpha,n} \geq (3 - \varepsilon)\sqrt{n\tan\frac{\alpha}{2}}) \to 1$.

3 Connectivity

Winkler [16] showed that $G_{\frac{\pi}{2},n}$ is connected and has diameter 3 w.h.p.. In this section we consider the case when $\alpha \neq \frac{\pi}{2}$. After some easy preliminary results, we address the case of α larger than $\frac{\pi}{2}$ and prove Theorem 1.2 in several parts. The following proposition is straightforward but will be useful in the proofs of our results.

Proposition 3.1. The area $|P^{\|}(x, y)|$ is maximized at $x = y = 0$ with

$$|P^{\|}(0, 0)| = \min\{1, 1 - \tan(\tfrac{\pi}{4} - \tfrac{\alpha}{2})\}.$$

Furthermore, if $P(x, y) \in Q \setminus (P^{\|}(0, 0) \cup P^{\|}(1, 1))$ and $x \leq y$, then

$$|P^{\|}(x, y)| = (x^2 + (1 - y)^2)(\tan(\tfrac{\pi}{4} + \tfrac{\alpha}{2}) - \tan(\tfrac{\pi}{4} - \tfrac{\alpha}{2}))/2 = (x^2 + (1 - y)^2)\tan\alpha.$$

$\qquad\square$

Theorem 3.2. *Each of the following occur with high probability,*

  (i) *$\mathcal{P}_{\alpha,n}$ contains a pair of incomparable elements if $\alpha = \pi - \omega(n^{-2})$,*

  (ii) *$\mathcal{P}_{\alpha,n}$ is a linear ordering (so the diameter of $G_{\alpha,n}$ is 1) if $\alpha = \pi - o(n^{-2})$,*

  (iii) *$\mathcal{P}_{\alpha,n}$ contains a pair of comparable elements if $\alpha = \omega(n^{-2})$,*

  (iv) *$\mathcal{P}_{\alpha,n}$ is an antichain if $\alpha = o(n^{-2})$.*

Proof. We prove (iii) and (iv), then (i) and (ii) follow by applying the coupling between $\mathcal{P}_{\alpha,n}$ and $\mathcal{P}_{\pi-\alpha,n}$ defined by the rotation by angle $\frac{\pi}{2}$ around the point $(\frac{1}{2}, \frac{1}{2})$.

W.l.o.g. we may assume that $\alpha = o(1)$. Let $X_{i,j}$ denote the event that $P_i, P_j \in \mathcal{P}_{\alpha,n}$ are comparable and let us write $X = \sum_{1 \leq i < j \leq n} X_{i,j}$ for the number of pairs of comparable elements. Clearly, we have

$$\mathbb{E}(X) = \binom{n}{2}\mathbb{E}(X_{i,j}) = \binom{n}{2}\int_0^1 \int_0^1 |P^{\|}(x, y)|\, dx dy.$$

By Proposition 3.1, $|P^{\|}(x,y)| = (x^2 + (1-y)^2)\tan\alpha$ except on a set of size at most $2|P^{\|}(0,0)|$ where it is at most $|P^{\|}(0,0)| = \Theta(\tan\alpha)$. Thus

$$\mathbb{E}(X_{i,j}) = 2\int\int_{0 \leq x < y \leq 1} (x^2 + (1-y)^2)\tan\alpha\, dxdy + O(|P^{\|}(0,0)|^2) = \Theta(\alpha).$$

Hence by the assumption $\alpha = o(1)$ we conclude that $\mathbb{E}(X) = \Theta(n^2\alpha)$. This gives that if $\alpha = o(n^{-2})$, then $\mathbb{E}(X) = o(1)$, which proves (iv), while if $\alpha = \omega(n^{-2})$ then $\mathbb{E}(X) \to \infty$. Note that if $\{i,j\} \cap \{i',j'\} = \emptyset$ then the random variables $X_{i,j}$ and $X_{i',j'}$ are independent. Furthermore, as Proposition 3.1 states $|P^{\|}(x,y)|$ is maximized at $|P^{\|}(0,0)|$, we have

$$\mathbb{E}(X_{i,j}X_{i,k}) = \mathbb{P}(X_{i,j} = 1)\mathbb{E}(X_{i,k} \mid X_{i,j} = 1) \leq \mathbb{E}(X_{i,j})|P^{\|}(0,0)|.$$

Therefore

$$\begin{aligned}
\mathrm{Var}(X) &= \sum_{i,j,i',j'} \left(\mathbb{E}(X_{i,j}X_{i',j'}) - \mathbb{E}(X_{i,j})\mathbb{E}(X_{i',j'})\right) \\
&\leq n\binom{n-1}{2}\mathbb{E}(X_{i,j})|P^{\|}(0,0)| + \binom{n}{2}\mathbb{E}(X_{i,j}) \\
&= O(n^3\alpha^2 + n^2\alpha).
\end{aligned}$$

Thus $\mathrm{Var}(X) = o((\mathbb{E}X)^2)$ and (iii) follows from Chebyshev's inequality. $\quad\square$

Theorem 3.3. *If $\alpha = \frac{\pi}{2} + \omega(n^{-1/2})$, then the diameter of $G_{\alpha,n}$ is at most 2.*

Proof. For any $0 < x \leq \frac{1}{2}$ let $Q_x^- = [0,x]^2$ and $Q_x^+ = [1-x,1]^2$. Note that for any $P \in Q \setminus Q_{1/2}^+$ we have $Q_x^+ \subseteq P^+$ if $(\frac{1}{2} - x)\tan\frac{\alpha-\pi/2}{2} \geq x$ and similarly for any $P \in Q \setminus Q_{1/2}^-$ we have $Q_x^- \subseteq P^-$ if $(\frac{1}{2} - x)\tan\frac{\alpha-\pi/2}{2} \geq x$. By the assumption on α we obtain that this inequality holds for some $x = \omega(n^{-1/2})$ which implies that both Q_x^+ and Q_x^- contain a point from $\mathcal{P}_{\alpha,n}$ w.h.p..

By these observations, if $P_1, P_2 \in Q \setminus Q_{1/2}^+$ or $P_1, P_2 \in Q \setminus Q_{1/2}^-$, then they have a common neighbor in $G_{\alpha,n}$ w.h.p., and if neither of the above hold, then $P_1 \in Q_{1/2}^+$, $P_2 \in Q_{1/2}^-$, and thus $P_2 \prec P_1$ or $P_2 \in Q_{1/2}^+$, $P_1 \in Q_{1/2}^-$ and thus $P_1 \prec P_2$. $\quad\square$

Theorem 3.2 (i) and Theorem 3.3 yield the following corollary.

Corollary 3.4. *If both $\alpha - \frac{\pi}{2} = \omega(n^{-1/2})$ and $\pi - \alpha = \omega(n^{-2})$ hold, then the diameter of $G_{\alpha,n}$ is 2 w.h.p..* $\quad\square$

The following theorem states that the lower bound of Theorem 3.3 is tight. Its proof is basically the same as Winkler's proof in [17] but to simplify the independency argument we work with the Poisson model.
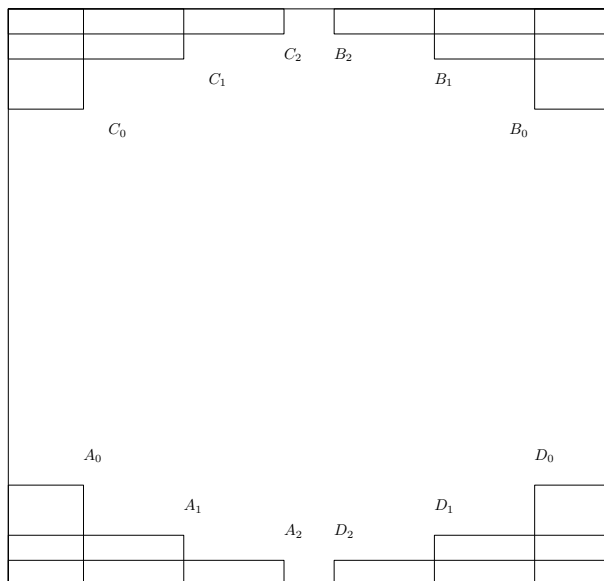
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Figure 1: Finding points at distance 3.

Theorem 3.5. *Let $\alpha = \frac{\pi}{2} + o(n^{-1/2})$. Then w.h.p. the diameter of $G_{\alpha,n}^P$ is 3.*

Proof. The fact that the diameter is at most 3 w.h.p. comes from Winkler's result that the diameter of $G_{\frac{\pi}{2},n}$ (and hence also $G_{\frac{\pi}{2},n}^P$) is 3 w.h.p.. Hence it is enough to show that w.h.p. there exist $P_1, P_2 \in \mathcal{P}_{\alpha,n}^P$ with $P_1^\| \cap P_2^\| \cap \mathcal{P}_{\alpha,n}^P = \emptyset$.

Write $\alpha = \frac{\pi}{2} + \frac{1}{\beta n^{1/2}}$ where $\beta = \beta_n \to \infty$. Fix an integer ℓ and construct rectangles $A_0, A_1, \ldots, A_\ell, B_0, B_1, \ldots, B_\ell, C_0, C_1, \ldots, C_\ell, D_0, D_1, \ldots, D_\ell$ as in Figure 1. Rectangles A_i, B_i, C_i, D_i will each have width $\beta^{\frac{i}{\ell}} n^{-1/2}$ and height $\beta^{-\frac{i}{\ell}} n^{-1/2}$, hence their area is $1/n$ and the probability that they contain a point in $\mathcal{P}_{\alpha,n}^P$ is $1 - 1/e$.

Let $J \subseteq Q$ denote the set of points that are contained in at least two of the rectangles defined above and let $A_i' = A_i \setminus J$, $B_i' = B_i \setminus J$, $C_i' = C_i \setminus J$, $D_i' = D_i \setminus J$. The area of the region $A_i \cap \bigcup_{j>i} A_j$ is $\beta_n^{-1/\ell} n^{-1}$, so the area of J is at most $4\ell \beta^{-1/\ell} n^{-1} = o(n^{-1})$. Thus $J \cap \mathcal{P}_{\alpha,n}^P = \emptyset$ w.h.p.. Note that for sufficiently large n, $\tan(\frac{\alpha}{2} - \frac{\pi}{4}) \leq \frac{1}{\beta n^{1/2}}$ and $\beta^{1/\ell} > 2$, so $\tan(\frac{\alpha}{2} - \frac{\pi}{4}) \leq \beta^{-(i-1)/\ell} n^{-1/2} - \beta^{-i/\ell} n^{-1/2}$ for all $i \leq \ell$. Therefore if say $P = (x,y) \in C_i'$ and $P' = (x',y') \in B_j'$ with $P \prec_\alpha P'$, then $y' > y - \beta_n^{-(i-1)/\ell} n^{-1/2} + \beta_n^{-i/\ell} n^{-1/2}$ and so $j \geq i-1$. Similarly, if $P \in D_i'$ and $P' \in B_j'$ with $P \prec_\alpha P'$ then $j \leq i+1$. Corresponding results hold for $P' \in A_j'$, so for any pair of points $P_1 \in C_i'$, $P_2 \in D_i'$ we have $(P_1^\| \cap P_2^\|) \setminus J \subseteq A_{i-1}' \cup A_i' \cup A_{i+1}' \cup B_{i-1}' \cup B_i' \cup B_{i+1}'$. Let E_j denote the event that $C_{3i+1}' \cap \mathcal{P}_{\alpha,n}^P$ and $D_{3i+1}' \cap \mathcal{P}_{\alpha,n}^P$ are not empty but $(A_{i-1}' \cup A_i' \cup A_{i+1}' \cup B_{i-1}' \cup B_i' \cup B_{i+1}') \cap \mathcal{P}_{\alpha,n}^P$ is empty. By the above, if n is large

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enough then any of the events $E_0, E_1, \ldots, E_{(\ell-1)/3}$ would provide us a pair of points whose distance in $G_{\alpha,n}$ is at least 3. As these events are independent, the probability that none of these events hold is at least $(1 - e^{-6}(1 - e^{-2})^2)^{\ell/3}$ for large n, which is smaller than any fixed $\varepsilon > 0$ provided $\ell > \ell(\varepsilon)$. $\qquad\square$

Corollary 3.6. *Let $\alpha = \frac{\pi}{2} + o(n^{-1/2})$. Then w.h.p. the diameter of $G_{\alpha,n}$ is 3.*

Proof. The fact that the diameter is at most 3 w.h.p. comes from Winkler's result that the diameter of $G_{\frac{\pi}{2},n}$ is 3 w.h.p.. Hence as above it is enough to show that w.h.p. there exist $P_1, P_2 \in \mathcal{P}_{\alpha,n}$ with $P_1^{\|} \cap P_2^{\|} \cap \mathcal{P}_{\alpha,n} = \emptyset$. Let X be a Poisson variable with mean $n + n^{2/3}$. Then by Theorem 3.5, w.h.p. there exist $P_1, P_2 \in \mathcal{P}_{\alpha,X}$ with $P_1^{\|} \cap P_2^{\|} \cap \mathcal{P}_{\alpha,X} = \emptyset$. But w.h.p. $X \geq n$, so by removing $X - n$ points of $\mathcal{P}_{\alpha,X}$ at random we obtain an instance of $\mathcal{P}_{\alpha,n}$. Clearly $P_1^{\|} \cap P_2^{\|} \cap \mathcal{P}_{\alpha,n} = \emptyset$, and, conditioned on X, with probability at most n^2/X^2 neither P_1 nor P_2 is removed. Since $n^2/X^2 = 1 - O(n^{-1/3})$ w.h.p., the result follows. $\qquad\square$

Now we turn to the case $\alpha < \frac{\pi}{2}$. The following definition is the key ingredient to understanding what can make $G_{\alpha,n}$ disconnected.

Definition 3.7. Let $\mathcal{P}$ be a finite partial suborder of Q equipped with the α-ordering. We say that $C \in Q$ (not necessarily in $\mathcal{P}$!) is a *cut point* of $\mathcal{P}$ if $C^{\|} \cap \mathcal{P} = \emptyset$.

We will need the following lemma.

Lemma 3.8. *Let $\mathcal{P}$ be a finite partial suborder of Q equipped with the α-ordering where $\alpha > 0$. Then its comparability graph G is disconnected if and only if at least one of the following the possibilities holds:*

(i) *$|\mathcal{P}| > 1$ and there exists $P \in \mathcal{P}$ such that P is a cut point of $\mathcal{P}$.*

(ii) *there exists $P_1, P_2 \in \mathcal{P}$ such that the intersection $C = (c_1, c_2)$ of $e_{\alpha,+,P_1}$ and $e_{\alpha,-,P_2}$ is a cut point of $\mathcal{P}$ and there exists points $P = (x, y)$ and $P' = (x', y')$ of $\mathcal{P}$ with $x - y < c_1 - c_2 < x' - y'$.*

Proof. Case (i) corresponds to an isolated vertex in the comparability graph G and a cut point satisfying (ii) defines a cut in G. To see the other direction of the lemma, let us suppose G is disconnected. If there exists an isolated vertex in G, then the corresponding element of $\mathcal{P}$ is a cut point and (i) holds. If there is no isolated vertex, then consider a "leftmost" component of G, i.e., one which contains an element $P = (x, y)$ of $\mathcal{P}$ with $x - y$ being minimal. Note that components can be ordered by the values of $x - y$ of their points, since if $P = (x, y)$ does not lie in a component $\mathcal{C}$ of G and if $\mathcal{C}$ contains points $P' = (x', y')$ and $P'' = (x'', y'')$ with

$x' - y' < x - y < x'' - y''$ then by considering a path from P' to P'' it must contain two such points that are comparable. But then either P' or P'' is comparable to P. Let P_1 (P_2) be the element in this component with $e_{\alpha,+,P_1}$ ($e_{\alpha,-,P_2}$) being closest to the point $(1,0)$. Let C be the intersection of $e_{\alpha,+,P_1}$ and $e_{\alpha,-,P_2}$. Then C is a cut point as if $P \in C^+ \cap \mathcal{P}$ then $P \succ_\alpha P_2$ but $e_{\alpha,+,P}$ lies to the right of $e_{\alpha,+,P_1}$, and similarly if $P \in C^- \cap \mathcal{P}$ then $P \prec_\alpha P_1$ but $e_{\alpha,-,P}$ lies to the right of $e_{\alpha,-,P_2}$. If $C = P_1$ or $C = P_2$ (and in particular, if $P_1 = P_2$) then C is an isolated vertex and (i) holds. By choice of leftmost component, any point $P = (x,y)$ in this component satisfies $x - y < c_1 - c_2$ and any point $P' = (x',y')$ not in this component satisfies $c_1 - c_2 < x' - y'$. Thus (ii) holds. $\qquad\square$

Proposition 3.9. If $\alpha = o(1)$ then $G_{\alpha,n}$ is disconnected w.h.p..

Proof. We claim that $C = (\frac{1}{\alpha^{1/3}\sqrt{n}}, 1 - \frac{1}{\alpha^{1/3}\sqrt{n}})$ is a cutpoint of $G_{\alpha,n}$. Indeed, by Proposition 3.1 we have $|C^\parallel| = \frac{2}{\alpha^{2/3}n}\tan\alpha = O(\frac{\alpha^{1/3}}{n}) = o(1/n)$ and thus $C^\parallel$ is empty w.h.p., while both squares $[0, \frac{1}{\alpha^{1/3}\sqrt{n}}] \times [1 - \frac{1}{\alpha^{1/3}\sqrt{n}}, 1]$ and $[\frac{1}{\alpha^{1/3}\sqrt{n}}, 1] \times [0, 1 - \frac{1}{\alpha^{1/3}\sqrt{n}}]$ are non-empty w.h.p.. Therefore $G_{\alpha,n}$ is disconnected. $\qquad\square$

Proposition 3.10. For any fixed α with $0 < \alpha < \frac{\pi}{2}$ there exist constants $0 < p_\alpha, p'_\alpha < 1$ such that
$$p_\alpha \leq \mathbb{P}(G^P_{\alpha,n} \text{ is connected}) \leq p'_\alpha.$$

Proof. The existence of p'_α follows from the fact that the point $C = (\frac{1}{\sqrt{n}}, 1 - \frac{1}{\sqrt{n}})$ is a cut point of $G^P_{\alpha,n}$ with positive probability.

We still have to prove a lower bound on the probability of $G^P_{\alpha,n}$ being connected. Write $t = \tan\frac{\alpha}{2}$ and let $x_1, x_2, \ldots, x_m$ be a geometric sequence with ratio $r = 1 + t$, $x_1 = \frac{1}{\sqrt{n}}$ and x_m the largest element of the sequence with $x_m < 1$. Define squares in the following way: for each integer $1 \leq i \leq m/2$ let $I_{2i-1} = [0, \frac{t}{2}x_{2i-1}] \times [1 - (1 + \frac{t}{2})x_{2i-1}, 1 - x_{2i-1}]$ and $I_{2i} = [x_{2i}, (1 + \frac{t}{2})x_{2i}] \times [1 - \frac{t}{2}x_{2i}, 1]$. Let U_i and V_i denote the top-left and bottom-right corner of I_i. Note that $e_{\alpha,-,V_{2i}}$ meets the line $x = 0$ at the point $P = (0, 1 - \frac{t}{2}x_{2i} - (1 + \frac{t}{2})x_{2i}\tan(\frac{\pi}{4} - \frac{\alpha}{2}))$. Now $\tan(\frac{\pi}{4} - \frac{\alpha}{2}) = (1 - t)/(1 + t)$ and $x_{2i} = (1+t)x_{2i-1}$ so $P = (0, 1 - \frac{1}{2}(t(1+t) + (2+t)(1-t))x_{2i-1}) = (0, 1 - x_{2i-1}) = U_{2i-1}$. By symmetry, $e_{\alpha,+,U_{2i+1}}$ meets V_{2i}, so I_{2i-1} and I_{2i+1} lie between $e_{\alpha,-,U_{2i}}$ and $e_{\alpha,+,V_{2i}}$. Hence $I_{2i-1}, I_{2i+1} \prec_\alpha I_{2i}$. Thus $I_1 \prec_\alpha I_2 \succ_\alpha I_3 \prec_\alpha I_4 \succ_\alpha \ldots$. Let J_i be the image of I_i when reflected to the line $x = y$, therefore we also have $J_1 \prec_\alpha J_2 \succ_\alpha J_3 \prec_\alpha J_4 \succ_\alpha \ldots$.

Let E_1 denote the event that all I_is and J_js are non-empty, let E_2 be the event that $\mathcal{P}^P_{\alpha,n}$ does not contain any point in $S = [0, x_2] \times [1 - x_1, 1] \cup [1 - x_1, 1] \times [0, x_2]$ and let E_3 denote the event that the points of $\mathcal{P}^P_{\alpha,n}$ in $R_\alpha \cup I_m \cup J_m$ lie in the same component of $G^P_{\alpha,n}$. The events E_1 and E_2 are independent as they involve disjoint areas. $\mathbb{P}(E_1) = \prod_{i=1}^m (1 - e^{-r^{2(i-1)}})^2 \geq \prod_{i=1}^\infty (1 - e^{-r^{2(i-1)}})^2 > 0$.
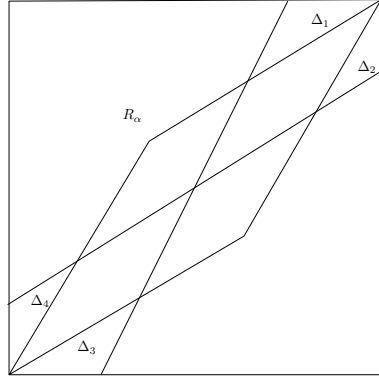
Figure 2: Connecting I_m to R_α.

Claim 3.11. $\mathbb{P}(E_3 \mid E_1) = 1 - o(1)$ and thus $\mathbb{P}(E_1 \cap E_2 \cap E_3)$ is larger than some p''_α that does not depend on n.

Proof of Claim. By applying Winkler's result and the linear transformation T^α we obtain that the points in R_α lie in the same component w.h.p.. The event E_1 implies that all points in $\bigcup_{i=1}^{m} I_i$ are contained in the same component and so are those in $\bigcup_{i=1}^{m} J_i$. Consider any point $P \in I_m \cup J_m$. Let $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ denote the 4 triangles forming $P^\|(\frac{1}{2}, \frac{1}{2}) \setminus R_\alpha$ and let R_1, R_2, R_3, R_4 denote the 4 parallelograms obtained by dividing R_α into 4 equal pieces similar to R_α (see Figure 2). Observe that, by the definition of m, $P^\|$ contains one of Δ_i which are of constant size, thus they are non-empty w.h.p., and so are the R_is. If $Q \in \Delta_i$, then $Q^\|$ contains one of the R_is, thus P is connected to a point of R_α by a path of length 2. $\qquad\square$

Claim 3.12. If E_1, E_2 and E_3 hold, then $G_{\alpha,n}^P$ is connected.

Proof of Claim. As E_2 and E_3 hold, all points lying in R_α and $\bigcup_{i=1}^{m}(I_i \cup J_i)$ belong to the same component. But $Q \setminus S \subseteq R_\alpha^\| \cup \bigcup_{i=1}^{m}(I_i^\| \cup J_i^\|)$. $\qquad\square$

Claim 3.11 and Claim 3.12 finishes the proof of the proposition. $\qquad\square$

Corollary 3.13. *For any fixed α with $0 < \alpha < \frac{\pi}{2}$ there exist constants $0 < p_\alpha, p'_\alpha < 1$ such that*

$$p_\alpha \leq \mathbb{P}(G_{\alpha,n} \text{ is connected}) \leq p'_\alpha.$$

Proof. As above, the existence of p'_α follows from the fact that the point $C = (\frac{1}{\sqrt{n}}, 1 - \frac{1}{\sqrt{n}})$ is a cut point of $G_{\alpha,n}$ with positive probability. For the lower bound, let X be a Poisson variable with mean $(1 - \varepsilon)n$ and consider $\mathcal{P}_{\alpha,n}$ as (w.h.p.) $\mathcal{P}_{\alpha,(1-\varepsilon n)}^P = \mathcal{P}_{\alpha,X}$

with $n - X$ additional points added uniformly at random. The proof of Proposition 3.10 shows that with probability at least p_α, $G_{\alpha,X}$ is connected and moreover that any point of $Q \setminus S$ is comparable to some point in $G_{\alpha,X}$. Thus adding points in $Q \setminus S$ will preserve connectivity. But the probability that any additional point lies in S is at most $|S|(n - X)$ which is $\leq \varepsilon$ w.h.p.. $\qquad\square$

Lemma 3.8 tells us that $G_{\alpha,n}$ is disconnected either because of the existence of an isolated vertex or a cutpoint defined by two points in $\mathcal{P}_{\alpha,n}$. To have some more insight on the probability distribution of the number of such points we prove the following two lemmas.

Lemma 3.14. *For any fixed α, $0 < \alpha < \frac{\pi}{2}$, let $Y(n,\alpha)$ denote the number of isolated vertices in $G_{\alpha,n}$. Then*

$$\lim_{n\to\infty} \mathbb{E}(Y(n,\alpha)) = \frac{\pi}{2\tan\alpha}.$$

Proof. The probability that the vertex corresponding to a point P_1 is isolated is given by

$$p := \int_0^1 \int_0^1 (1 - |P^\|(x,y)|)^{n-1}\, dx dy.$$

For large n, the square $[0, \frac{\log n}{\sqrt{n}}] \times [1 - \frac{\log n}{\sqrt{n}}, 1]$ is disjoint from $(P^\|(0,0) \cup P^\|(1,1))$ and thus by Proposition 3.1 for points lying in $[0, \frac{\log n}{\sqrt{n}}] \times [1 - \frac{\log n}{\sqrt{n}}, 1]$ the area $|P^\|(x, 1-y)|$ is $(x^2 + y^2)\tan\alpha$ and for points in $[1 - \frac{\log n}{\sqrt{n}}, 1] \times [0, \frac{\log n}{\sqrt{n}}]$ we have $|P^\|(1-x, y)| = (x^2 + y^2)\tan\alpha$. For points outside these squares the area $|P^\|(x,y)|$ is at least $\Omega\left(\frac{(\log n)^2}{n}\right)$, thus

$$p = 2\int_0^{\frac{\log n}{\sqrt{n}}} \int_0^{\frac{\log n}{\sqrt{n}}} (1 - (x^2 + y^2)\tan\alpha)^{n-1}\, dx dy + n^{-\Omega(\log n)}$$

$$= 2\int_0^{\frac{\log n}{\sqrt{n}}} \int_0^{\frac{\log n}{\sqrt{n}}} e^{-n(x^2+y^2)\tan\alpha}\, dx dy + n^{-\Omega(\log n)},$$

where we have used the fact that $e^{-nz} = (1 - z + O(z^2))^n = (1-z)^n + O(nz^2)$ for $z < 1$, and hence $(1-z)^{n-1} = e^{-nz} + O(z + nz^2) = O((\log n)^4/n)$ when $z = O((\log n)^2/n)$. Writing $u = \sqrt{n\tan\alpha}\, x$, $v = \sqrt{n\tan\alpha}\, y$ we obtain

$$p = \frac{2}{n\tan\alpha} \left(\int_0^{\sqrt{\tan\alpha}\log n} e^{-u^2}\, du\right)^2 + n^{-\Omega(\log n)} = \frac{\pi}{2n\tan\alpha} + n^{-\Omega(\log n)}.$$

The result now follows since $\mathbb{E}(Y(n,\alpha)) = np$. $\qquad\square$

Let us define the following random variables: $Z_{i,j,u,\alpha}$ $(Z_{i,j,d,\alpha})$ is the indicator variable of the event that both $P_i, P_j \in \mathcal{P}_{\alpha,n}$ lie in the half-plane $y \geq x$ $(y \leq x)$, the point $C = e_{\alpha,+,P_i} \cap e_{\alpha,-,P_j}$ is a cut point of $\mathcal{P}_{\alpha,n}$ and C is closer to the line $x = y$ than P_i and P_j. Note that the events associated with $Z_{i,j,u,\alpha}$ and $Z_{j,i,u,\alpha}$ are not the same. By the result of Winkler [16] and applying the linear transformation T^α defined in Section 2, we know that the points of $\mathcal{P}_{\alpha,n}$ lying in the rhombus R_α form a connected component. Lemma 3.8 shows that the sum of the random variables $Z_{i,j,u,\alpha}$, $Z_{i,j,d,\alpha}$ is the number of components that contain more than one vertex and that are different from the component containing the points in R_α. If we denote the sum by $Z(n,\alpha)$, then we obtain the following proposition.

Proposition 3.15. $G_{\alpha,n}$ is connected if and only if $Y(n,\alpha) + Z(n,\alpha) = 0$. $\qquad\square$

Lemma 3.16. *For any $\alpha < \frac{\pi}{2}$ let us write $t_\alpha = \tan(\frac{\pi}{4} - \frac{\alpha}{2})$. Then*

$$\lim_{n\to\infty} \mathbb{E}(Z(n,\alpha)) = \frac{t_\alpha}{\tan\alpha}.$$

Proof. Let us compute $\mathbb{E}(Z_{i,j,u,\alpha}) = \mathbb{P}(Z_{i,j,u,\alpha} = 1)$. The intersection point $C = (x, 1-y)$ of $e_{\alpha,+,P_i}$ and $e_{\alpha,-,P_j}$ determines two line segments on which P_i and P_j should lie. Allowing the these lines to vary by distances du and dv perpendicular to themselves causes these lines to trace out two infinite strips S_1 and S_2 of widths du and dv respectively. Now fix these strips and allow P_i, P_j, and C to move. If P_i lies in S_1 and P_j lies in S_2 then C lies in $S_1 \cap S_2$, and conversely, if C lies in $S_1 \cap S_2$ then P_i lies in S_1 and P_j lies in S_2. The intersection of S_1 and S_2 is a parallelogram of area $du\,dv/\sin\alpha$. Requiring P_i to be to the left of C and P_j to be above C restricts these points to regions of area approximately $x\,dv/\cos(\frac{\pi}{4} - \frac{\alpha}{2})$ and $y\,du/\cos(\frac{\pi}{4} - \frac{\alpha}{2})$ respectively. Thus we can estimate the probability that C lies in the parallelogram $S_1 \cap S_2$ given a random choice of P_i and P_j as $xy\,du\,dv/\cos^2(\frac{\pi}{4} - \frac{\alpha}{2})$ and hence the probability density function of the location of C is on average about

$$\frac{xy\,du\,dv/\cos^2(\frac{\pi}{4} - \frac{\alpha}{2})}{du\,dv/\sin\alpha} = \frac{xy\sin\alpha}{\cos^2(\frac{\pi}{4} - \frac{\alpha}{2})} = 2xyt_\alpha\tan\alpha$$

in a small parallelogram near $(x, 1-y)$. Here we have used the identity $2t_\alpha\cos^2(\frac{\pi}{4} - \frac{\alpha}{2}) = 2\sin(\frac{\pi}{4} - \frac{\alpha}{2})\cos(\frac{\pi}{4} - \frac{\alpha}{2}) = \sin(\frac{\pi}{2} - \alpha) = \cos\alpha$ to simplify the trigonometric expression. Thus we have

$$\mathbb{P}(Z_{i,j,u,\alpha} = 1) = 2t_\alpha\tan\alpha \iint_{x+y<1} xy(1 - |C^{\|}(x, 1-y)|)^{n-2}\,dx\,dy.$$

Using Proposition 3.1 as in the proof of Lemma 3.14, this becomes

$$2t_\alpha\tan\alpha \int_0^{\frac{\log n}{\sqrt{n}}} \int_0^{\frac{\log n}{\sqrt{n}}} xye^{-n(x^2+y^2)\tan\alpha}\,dx\,dy + n^{-\Omega(\log n)}.$$

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Writing again $u = \sqrt{n \tan \alpha}\, x$, $v = \sqrt{n \tan \alpha}\, y$ we obtain that

$$\mathbb{E}(Z_{i,j,u,\alpha}) = 2t_\alpha \tan \alpha \left(\frac{1}{n \tan \alpha} \int_0^{\sqrt{\tan \alpha}\, \log n} u e^{-u^2}\, du \right)^2 + n^{-\Omega(\log n)}$$

$$= \frac{t_\alpha}{2n^2 \tan \alpha} + n^{-\Omega(\log n)}.$$

The result follows as $\mathbb{E}(Z(n, \alpha)) = n(n-1)\mathbb{E}(Z_{i,j,u,\alpha}) + n(n-1)\mathbb{E}(Z_{i,j,d,\alpha})$. $\qquad \square$

By Lemma 3.14 and Lemma 3.16 we obtain the following corollary.

Corollary 3.17. *If $\mathfrak{P}$ denotes the property of having a connected comparabilty graph, then*

$$\tau(\mathfrak{P}) = \tfrac{\pi}{2}.$$

There are many examples for hitting-time results in the Erdős-Rényi random graph process that link some monotone graph property to the minimum degree. One of the earliest and most well-known such result is due to Bollobás and Thomason [6]: whenever the minimum degree becomes 1, the random graph gets connected. We finish this section with a result which shows that the evolution of $G_{\alpha,n}$ and $G^P_{\alpha,n}$ is very much different from that of the Erdős-Rényi graph.

Proposition 3.18. *For any fixed $\alpha < \frac{\pi}{2}$ there exists a constant $p''_\alpha > 0$ such that*

$$\mathbb{P}(G^P_{\alpha,n} \text{ is disconnected but contains no isolated vertices}) > p''_\alpha.$$

Proof. We will use the ideas and definitions of Proposition 3.10. Let E'_2 denote the event that $\mathcal{P}^P_{\alpha,n}$ does not contain any point in $S' = ([0, x_2] \times [1 - x_1, 1] \cap (P^\parallel(0, 1 - x_1) \cup P^\parallel(x_2, 1)) \cup [1 - x_1, 1] \times [0, x_2]$ and contains exactly 2 comparable points in $S'' = ([0, x_2] \times [1 - x_1, 1] \setminus (P^\parallel(0, 1 - x_1) \cup P^\parallel(x_2, 1))$. If E_1, E_3 are as defined in Proposition 3.10, then as in the proof of Proposition 3.10 $E_1 \cap E'_2 \cap E_3$ implies that $G^P_{\alpha,n}$ contains 2 connected components one of which consists of the 2 points of $\mathcal{P}^P_{\alpha,n} \cap S''$. The events E_1 and E'_2 are independent as they involve disjoint subsets of Q and their probability is bounded away from 0. Thus we are done by Claim 3.11. $\qquad \square$

Note that a similar statement for $\mathcal{P}_{\alpha,n}$ holds with a proof identical to that of Corollary 3.13.

4 Concluding remarks and open problems

Theorem 1.1 states that the size of the largest chain in $\mathcal{P}_{\alpha,n}$ is asymptotically $2\sqrt{n\tan\frac{\alpha}{2}}$ provided α does not tend to π or to 0 too quickly. We would be interested to know what can one say when the conditions of Theorem 1.1 are not satisfied. Note that Theorem 3.2 settles the case of extremely quickly convergent α as the proof of its part (i) shows that if $\pi - \alpha = \omega(n^{-2})$, then $L_{\alpha,n} \leq n - K$ for any constant K. The threshold function of the existence of a chain of length r cannot be deduced directly from Theorem 1.1 (iii), but a very similar proof works of which we give a sketch here.

Theorem 4.1. *For any fixed $r \geq 2$ we have*

$$\lim_{n\to\infty} \mathbb{P}(L_{\alpha,n} < r) = \begin{cases} 1 & \text{if } \alpha = o\left(n^{-\frac{r}{r-1}}\right), \\ 0 & \text{if } \alpha = \omega\left(n^{-\frac{r}{r-1}}\right). \end{cases}$$

Proof. W.l.o.g we may assume that $\alpha = o(1)$. Let $X_{i_1,\ldots,i_r}$ denote the indicator function of the event that $P_{i_1} \prec_\alpha P_{i_2} \prec_\alpha \cdots \prec_\alpha P_{i_r}$ holds. As in the α-ordering we have $|P^+(x,y)| = \Theta((\min\{1-x,1-y\})^2\alpha)$, so we see that $p_r := \mathbb{E}(X_{i_1,\ldots,i_r}) = \Theta(\alpha^{r-1})$. Thus the expected number of chains of length r is $\mathbb{E}(X_{\alpha,r,n}) = \Theta(n^r\alpha^{r-1})$ from which the first statement follows.

Clearly, if $i_1,\ldots,i_r$ and $i'_1,\ldots,i'_r$ are all distinct, then $X_{i_1,\ldots,i_r}$ and $X_{i'_1,\ldots,i'_r}$ are independent and $\mathbb{E}(X_{i_1,\ldots,i_r}X_{i'_1,\ldots,i'_r}) = p_r^2$. If $i_1,\ldots,i_r$ and $i'_1,\ldots,i'_r$ have $j \geq 1$ elements in common, then $\mathbb{E}(X_{i'_1,\ldots,i'_r} \mid X_{i_1,\ldots,i_r} = 1) = O(\alpha^{r-j})$. Therefore for these pairs of index sequences we have $\mathbb{E}(X_{i_1,\ldots,i_r}X_{i'_1,\ldots,i'_r}) = O(\alpha^{2r-1-j})$. Summing over all possible j we obtain

$$\mathbb{E}(X_{\alpha,r,n}^2) = n_{(2r)}p_r^2 + \sum_{j=1}^{r} O(n^{2r-j}\alpha^{2r-1-j}) = (1+o(1))(n_{(r)}p_r)^2 = (1+o(1))(\mathbb{E}(X_{\alpha,r,n}))^2,$$

where $n_{(r)} = \frac{n!}{(n-r)!}$ and we used $\alpha = \omega\left(n^{-\frac{r}{r-1}}\right)$ for the second equality. The theorem follows from Chebyshev's inequality. $\square$

Concerning the connectivity of $G_{\alpha,n}$, the most important open problem is to determine the limiting probability $\lim_{n\to\infty} \mathbb{P}(G_{\alpha,n}$ is connected$)$ for any fixed $0 < \alpha < \frac{\pi}{2}$. Another problem that remains open is to calculate the diameter of $G_{\alpha,n}$ when $\alpha = \frac{\pi}{2} - \beta$ with $0 < \beta = o(1)$. Note that Corollary 3.17 states that in this case $G_{\alpha,n}$ is connected w.h.p..

References

[1] D. Aldous, P. Diaconis, Longest increasing subsequences: from patience sorting to the Baik-Deift-Johansson theorem, *Bull. Amer. Math. Soc.* **36** (1999), 413–432.

[2] J. Baik, P. Deift, K. Johansson, On the distribution of the length of the longest increasing sequence of random permutations, *J. Amer. Math. Soc.* **12** (1999), 1119–1178.

[3] B. Bollobás, G. Brightwell, The height of a random partial order: concentration of measure, *The Annals of Applied Probability* **2** (1992), 1009–1018.

[4] B. Bollobás, G. Brightwell, Box spaces and random partial orders, *Trans. Amer. Math. Soc.* **324** (1991), 347–353.

[5] B. Bollobás, S. Janson, On the Length of the Longest Increasing Subsequence in a Random Permutation, *Combinatorics, geometry and probability* (Cambridge, 1993), 121–128, Cambridge Univ. Press, Cambridge, 1997.

[6] B. Bollobás, A. Thomason, Random graphs of small order, Random graphs '83 (Poznan, 1983), North-Holland Math. Stud., Vol. 118, North-Holland, Amsetrdam, 1985, pp. 47–97.

[7] B. Bollobás, P. Winkler, The longest chain among random points in Euclidean space, *Proc. Amer. Math. Soc.* **103** (1988), 347–353.

[8] P. Erdős, A. Rényi, On the evolution of random graphs, Publications of the Mathematical Institute of the Hungarian Academy of Sciences **5, 17–61.** (1960)

[9] G. Brightwell, Models of random partial orders, *Surveys in combinatorics,* 1993 (Keele), 53–83, London Math. Soc. Lecture Note Ser., 187, Cambridge Univ. Press, Cambridge, 1993.

[10] A. Frieze, On the length of the longest monotone subsequence in a random permutation, *Ann. Appl. Probab.* **1** (1991), 301–305.

[11] J.M. Hammersley, A few seedlings of research, *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, 345–394. Univ. California Press, Berkeley, Calif., 1972.

[12] J.H. Kim, On increasing subsequences of random permutations, *J. Combin. Theory Ser. A* **76** (1996), 148–155.

[13] B.F. Logan, L.A. Shepp, A variational problem for random Young tableaux, *Advances in Math.* **26** (1977), 206–222.

[14] S.M. Ulam, Monte Carlo calculations in problems of mathematical physics, Modern mathematics for the engineer: Second series, pp. 261–281 McGraw-Hill, New York, 1961.

[15] A.M. Vershik, S.V. Kerov Asymptotic behavior of the Plancherel measure of the symmetric group and the limit form of Young tableaux. (Russian) *Dokl. Akad. Nauk SSSR* **233** (1977), 1024–1027.

[16] P. Winkler, Random orders, *Order* **1** (1985), 317–335.

[17] P. Winkler, Connectedness and diameter for random orders of fixed dimension, *Order* **2** (1985), 165–171.