Vertex Distinguishing Colorings of Graphs with $\Delta(G) = 2$

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Abstract

In a paper by Burris and Schelp [3], a conjecture was made concerning the number of colors $\chi'_s(G)$ required to proper edge-color G so that each vertex has a distinct set of colors incident to it. We consider the case when $\Delta(G) = 2$, so that G is a union of paths and cycles. In particular we find the exact values of $\chi'_s(G)$ and hence verify the conjecture when G consists of just paths or just cycles. We also give good bounds on $\chi'_s(G)$ when G contains both paths and cycles.

1 Introduction

A proper edge-coloring of a simple graph G is called *vertex-distinguishing* if for any two distinct vertices u and v in G, the set of colors assigned to the edges incident to u differs from the set of colors incident to v. A vertex-distinguishing proper edge-coloring is also called a *strong* coloring. A graph is *vertex-distinguishing edge-colorable* or a *vdec*-graph, if it contains no more than one isolated vertex and no isolated edges. Clearly, a graph has a strong coloring if and only if it is a vdec-graph. The minimal number of colors required for a strong coloring of G is denoted by $\chi'_s(G)$. This concept of strong coloring was introduced independently by Burris and Schelp [3], and (for non-proper colorings) by Aigner, Triesch, and Tuza [1]. A similar concept was discussed in [5]. Other articles involving such colorings appear in [4–9].

Let $n_d = n_d(G)$ denote the number of vertices of degree d in a vdec-graph G. It is clear that $\binom{\chi'_s(G)}{d} \geq n_d$ for all d with $\delta(G) \leq d \leq \Delta(G)$. The conjecture given by Burris and Schelp [3] is as follows.

Conjecture 1 *Let G be a vdec-graph and let k be the minimum integer such that $\binom{k}{d} \geq n_d$ for all d such that $\delta(G) \leq d \leq \Delta(G)$. Then $\chi'_s(G) = k$ or $k + 1$.*

The conjecture appears to be difficult even when the graph G is regular. It was shown by Aigner, Triesch and Tuza [1] that if G is 2-regular of order n then it has a (not necessarily proper) vertex-distinguishing edge-coloring with at most $\frac{9}{2}\sqrt{2n}$ colors. One of our aims in this paper is to improve this bound to one that is close to best possible (see Corollary 6). Recently Černý, Horňák, and

Soták [4] determined the exact value of $\chi'_s(G)$ when G is a path or cycle; this had been done independently by Burris [9].

In this paper we shall consider the case when $\Delta(G) = 2$. The case of larger Δ is much harder and the methods described here do not seem to be applicable. For a graph of maximal degree 2, the vertex-distinguishing coloring problem can be translated into a problem of packing the line graph $L(G)$ of G into a complete graph, so most of this paper is about such packings.

As usual, we write K_n for the complete graph, E_n for the empty graph and C_n for a cycle on n vertices. If we have a specific set S of vertices in mind, we shall also use notations such as K_S and E_S. Write P_n for a path of length n (on $n+1$ vertices) and $P(v_1, v_2, \dots, v_r)$ for the *trail* of length $r-1$ on the vertices v_i with edges $v_i v_{i+1}$. We do *not* require the v_i to be distinct. For any two graphs G_1 and G_2, write $G_1 \cup G_2$ for the vertex disjoint union of G_1 and G_2.

If G_1 and G_2 are graphs, a *packing* of G_1 into G_2 is a map $f \colon V(G_1) \to V(G_2)$ such that $xy \in E(G_1)$ implies $f(x)f(y) \in E(G_2)$ and the induced map on edges $xy \mapsto f(x)f(y)$ is a injection from $E(G_1)$ to $E(G_2)$. We do *not* require f to be injective on vertices, so if G_1 contains a cycle or path, its image in G_2 will be a circuit (closed trail) or trail. We shall call a packing *exact* if the packing induces a bijection between $E(G_1)$ and $E(G_2)$. We shall write $G_1 \mapsto G_2$ to mean that an exact packing of G_1 into G_2 exists.

In section 2 we shall consider the case when G is a union of cycles $C_{m_1} \cup \dots \cup C_{m_t}$. In this case the line graph $L(G)$ is also of the form $C_{m_1} \cup \dots \cup C_{m_t}$. If G is given a strong coloring by n colors, then we get a packing of $L(G)$ as t edge-disjoint circuits in K_n. Each edge of G corresponds to a vertex of $L(G)$ which is mapped to a color (vertex) of K_n. Conversely if we have a packing of $L(G)$ into K_n then we can color each edge of G with the image of the corresponding vertex of $L(G)$ in K_n. Since the edges of $L(G)$ are mapped to distinct edges in K_n, the resulting coloring on G is strong. Thus the exact value of $\chi'_s(G)$ is just the smallest n such a packing of $L(G)$ into K_n exists. We have therefore reduced the problem to one of packing a union of cycles into K_n.

In section 3 we prove the conjecture in the case when G is a union of p paths $P_{l_1+1} \cup \dots \cup P_{l_p+1}$. Since G is a vdec-graph, we can assume $l_i \geq 1$. As before, a strong coloring of G is equivalent to a certain packing of the line graph $L(G)$ into K_n. The line graph is a disjoint union of paths $P_{l_1} \cup P_{l_2} \cup \dots \cup P_{l_p}$, where each path is of length one less than the corresponding path of G. In this case the existence of a strong coloring of G with n colors is equivalent to the existence of a packing of $L(G)$ into K_n with the extra condition that we require the $2p$ endpoints of the paths to be mapped to distinct vertices in K_n.

In section 4 we consider the general case when G is a vdec-graph with $\Delta(G) = 2$. In this case G is a union of paths (of lengths at least two), cycles, and possibly a single isolated vertex. The presence or absence of an isolated vertex has no effect on the coloring, so we can ignore it. Once again, the existence of a strong coloring of G with n colors is equivalent to the existence of a packing of the line graph into K_n with the endpoints of the paths mapped to distinct vertices of K_n. The result we prove is slightly weaker in this case and we do not obtain the exact values of $\chi'_s(G)$.

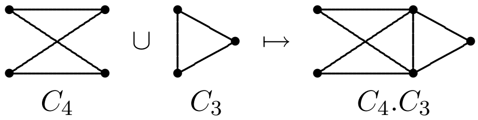

Figure 1: Example of a linking of two graphs

2 Unions of Cycles

Write $G_1.G_2$ for an edge-disjoint union of two graphs which is *not* disjoint on vertices. In other words, $G_1.G_2$ is the image of $G_1 \cup G_2$ under a packing which is injective on $V(G_1)$ and injective on $V(G_2)$, but in which some vertices of G_1 are identified with some vertices of G_2. Whenever we use the notation $G_1.G_2$, we shall make it clear which pairs of vertices are identified. Vertices of G_i that are identified will sometimes be called a *link* of G_i, and we shall call the identification a *linking* of G_1 and G_2.

Assume that $C_a.C_b$ is obtained by linking cycles C_a and C_b at at least one vertex. We can pack C_{a+b} into $C_a.C_b$ by picking such a link vertex v and going round C_a starting at v, then going round C_b. By induction, if we have a sequence of linked cycles $C_{a_1}.C_{a_2}\dots C_{a_t}$ with each meeting the next in at least one vertex, we can pack any cycle of length $\sum_{i=1}^{t} a_i$ into such a graph. We shall use this observation many times in what follows.

For $n \equiv 1, 3 \bmod 6$, there exist Steiner triple systems that pack K_n with $\frac{n}{6}(n-1)$ triangles. If we have a such a packing, then each edge belongs to a unique triangle. We can define a *trail* of triangles as a sequence of triangles determined by a trail (of edges) in which each edge belongs to a distinct triangle. The existence of a trail of triangles is stronger than the existence of a linked sequence of triangles $T_1.T_2\dots T_t$. Indeed, for such a sequence to form a trail of triangles we need $V(T_i) \cap V(T_{i+1}) = \{v_i\}$ with $v_i \neq v_{i+1}$.

Lemma 1 *If $n \equiv 1$ or $3 \bmod 6$ then we can pack K_n with a trail of triangles of length at least $\frac{n}{6}(n-1) - 1$.*

Proof. Let $\mathcal{S} = \{T_1, \dots, T_N\}$ be a Steiner Triple System for K_n. Pick one triangle, T_1, say. Let T_1 have vertex set $V(T_1) = \{r_1, r_2, r_3\}$ and let $M = V(K_n) \setminus V(T_1)$ be the set of the remaining $n-3$ vertices of K_n. Let S_M be the subset of triangles $T_i \in \mathcal{S}$ that have all their vertices in M. Each vertex $v \in M$ meets precisely three triangles that are not in S_M, one for each edge vr_j. Hence each $v \in M$ is incident to exactly $n-7$ edges that are in triangles in S_M. Let S be a subset of S_M and let m be the number of vertices in M meeting some triangle in S. The number of edges of triangles in S is $3|S| \leq \frac{m}{2}(n-7)$ and so by Hall's marriage theorem, we can assign triangles $T_i \in S_M$ to vertices v_i, such that $v_i \in V(T_i)$ and no more than $\lceil \frac{n-7}{6} \rceil$ triangles are assigned to each vertex of M.

Construct a subgraph G of K_n consisting of one edge in M from each triangle T_i, $i \neq 1$. For each triangle $T_i \in S_M$ we let G contain the unique edge of T_i that does not meet v_i. For the other triangles we let G contain the unique edge of T_i which lies in M. Each vertex $v \in M$ has degree in G of at least $\frac{n-1}{2} - \lceil \frac{n-7}{6} \rceil \geq \frac{n}{3}$ since there are $\frac{n-1}{2}$ triangles of $\mathcal{S}$ that meet v and each triangle that meets v other than those with $v_i = v$ contributes one to this degree.

We will now modify G so as to make all the vertices have even degrees. Let I_j, $j = 1, 2, 3$ be the graph containing the edges in M of the triangles meeting r_j. Clearly I_j are 1-factors of $G[M]$. Let $C = C(u_1, u_2, \ldots, u_r)$ be a component cycle of $I_1 \cup I_2$ in G. For each vertex u_i in turn, if the degree in G of u_i is odd, replace the edge $u_i u_{i+1}$ in G by the edge $u_{i+1} r_1$ or $u_{i+1} r_2$ of the triangle containing it. Do this for each cycle in turn. The resulting graph G' has even degree at each vertex of M and contains one edge from each triangle T_i, $i \neq 1$. If the degree of r_1 (and hence r_2) in G' is odd, add the edge $r_1 r_2$ of the triangle T_1. Otherwise discard T_1. The graph now has even degree at all vertices. The degree in G' of any vertex in M is at least $\frac{n}{3} - 1$, so any component of G' meeting M must have at least $\frac{n}{3}$ vertices and does not meet r_3. Hence the graph G' has at most two (non-singleton) components. If it has two components, then pick an edge uv connecting them. Removing an edge from either component does not increase the number of components (each component has an Eulerian circuit), hence adding uv and removing the edge in G' that belonged to its triangle gives a new graph G' which is connected (apart from isolated vertices) and has even degree at all except possibly two vertices. It therefore has an Eulerian trail. This gives a trail of triangles in K_n which includes all except at most one triangle T_1. □

In fact with more work it is possible to improve Lemma 1 to include all the triangles of the Steiner triple system, but we shall not need that here.

We shall define for some graphs *initial* and *final* links as (ordered) pairs of vertices, (possibly the same pair). In these cases $G_1.G_2$ will identify the final link of G_1 with the initial link of G_2 (in the same order). The graph $G_1.G_2$ will be undefined if an edge occurs in both these links. The initial link of the resultant graph will be that of G_1 and the final link will be that of G_2. This makes . into an associative operation on such graphs when defined. Similarly, the initial link of $G_1 \cup G_2$ will be that of G_1 and the final link will be that of G_2. We shall also write $G^{\cdot n}$ for $G.G \ldots G$ and $G^{\cup n}$ for $G \cup \ldots \cup G$ where there are n copies of G.

Write $G_1 + G_2$ for the join of G_1 and G_2, i.e., the graph $G_1 \cup G_2$ with all edges connecting G_1 and G_2 included. Define O to be the graph of an octahedron, so $O = K_{2,2,2} = E_2 + E_2 + E_2$. The first E_2 will be the initial link and the last E_2 will be the final link of O. In fact by symmetry it does not matter which E_2's are chosen, or the order of the vertices in either link. We shall now pack K_{2n} with trails of octahedra.

Lemma 2 *If* $n \equiv 1$ *or* $3 \bmod 6$, *there is a packing of* $O^{\cdot a}$ *into* K_{2n} *with* $a \geq \frac{n}{6}(n-1) - 1$.

Proof. Pack K_n with a trail of triangles using Lemma 1. Now replace each vertex v of K_n by a pair of vertices v_0, v_1, and each edge uv by four edges $u_i v_j$. The resulting graph is just K_{2n} with a 1-factor removed. The triangles become octahedra and a trail of triangles becomes a packing of linked octahedra $O^{\cdot a}$. The result follows. □

For a path P_n of length n with endpoints u and v, make (u, v) both the initial and final link of P_n. Write $C'_n = C_n \cup E_1$ to denote a cycle of length n together with an extra independent vertex. The pair (u, v) will be the initial and final link of C'_n where u is the independent vertex and v is any other vertex. The graph $P_{a_1,\ldots,a_r} = P_{a_1}.P_{a_2} \ldots P_{a_r}$ will be a graph with specified link vertices (u, v) consisting of independent paths of length a_i from u to v. In the special case when $r = 0$ we write $P_\emptyset$ for the empty graph E_2 on $\{u, v\}$. We write $S_{a,b,c,d}$ for a cycle with initial link (u, v), final link (u', v') and four independent paths connecting them. A path of length a connects u and

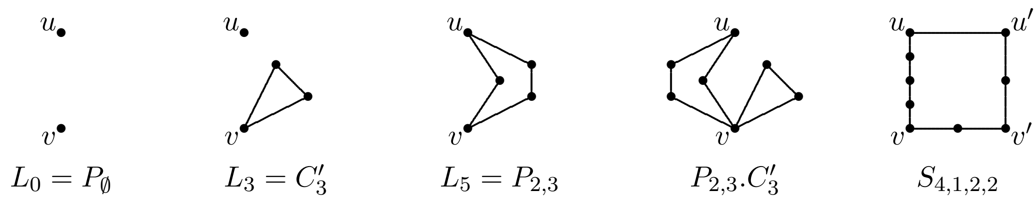

Figure 2: Examples of L_n and $S_{a,b,c,d}$.

v, a path of length b connects u and u', a path of length c connects v and v', and a path of length d connects u' and v'.

Definition. The graphs L_n are defined as

$$L_0 = P_\emptyset, \quad L_3 = C_3', \quad L_4 = P_{2,2}, \quad L_5 = P_{2,3}, \quad \text{and} \quad L_n = P_{4,n-4} \text{ for } n \geq 7.$$

L_6 will be defined as *either* $P_{3,3}$ *or* $P_{4,2}$. Note that we can pack C_n' exactly into L_n for all $n > 0$ with initial link matching.

Lemma 3 *The following graphs can all be packed into O with initial and final links matching:*

$$L_3.C_3' \cup C_3.L_3, \qquad P_{2,2,2,2} \cup P_{2,2}, \qquad P_\emptyset \cup P_{3,3,3,3}, \qquad S_{4,1,1,3}.C_3', \qquad S_{4,1,2,2}.C_3',$$
$$L_n.C_3' \cup L_{9-n}, \ (4 \leq n \leq 6) \qquad \textit{and} \qquad L_n \cup L_{12-n}, \ (3 \leq n \leq 9).$$

Proof. In each of the listed graphs all the links are uniquely specified by the rules given above except for the link $C_3.L_3$ in the first graph. For this we just claim there is some linking that will do. Number the vertices of O from 0 to 5 so that $O = E_{\{0,1\}} + E_{\{2,3\}} + E_{\{4,5\}}$ with $(0,1)$ the initial link and $(4,5)$ the final link. We pack the paths as follows:

$$\begin{aligned}
L_3.C_3' \cup C_3.L_3 &\mapsto \{P(1,2,4,1); P(1,3,5,1); P(4,3,0,4); P(5,2,0,5)\}\\
P_{2,2,2,2} \cup P_{2,2} &\mapsto \{P(0,2,1), P(0,3,1), P(0,4,1), P(0,5,1); P(4,2,5), P(4,3,5)\}\\
P_\emptyset \cup P_{3,3,3,3} &\mapsto \{; P(4,0,2,5), P(4,2,1,5), P(4,1,3,5), P(4,3,0,5)\}\\
S_{4,1,1,3}.C_3' &\mapsto \{P(0,2,4,3,1), P(0,4), P(1,5), P(4,1,2,5); P(5,0,3,5)\}\\
S_{4,1,2,2}.C_3' &\mapsto \{P(0,2,4,3,1), P(0,4), P(1,2,5), P(4,1,5); P(5,0,3,5)\}\\
P_{4,2}.C_3' \cup C_3' &\mapsto \{P(0,2,4,3,1), P(0,4,1); P(1,2,5,1); P(5,0,3,5)\}\\
P_{3,3}.C_3' \cup C_3' &\mapsto \{P(0,4,3,1), P(0,2,4,1); P(1,2,5,1); P(5,0,3,5)\}\\
P_{2,3}.C_3' \cup P_{2,2} &\mapsto \{P(0,3,1), P(0,2,4,1); P(1,2,5,1); P(4,3,5), P(4,0,5)\}\\
P_{2,2}.C_3' \cup P_{2,3} &\mapsto \{P(0,3,1), P(0,4,1); P(1,2,5,1); P(4,3,5), P(4,2,0,5)\}\\
P_{4,5} \cup C_3' &\mapsto \{P(0,2,4,3,1), P(0,4,1,2,5,1); P(5,0,3,5)\}\\
P_{4,4} \cup P_{2,2} &\mapsto \{P(0,2,4,3,1), P(0,3,5,2,1); P(4,1,5), P(4,0,5)\}\\
P_{4,3} \cup P_{3,2} &\mapsto \{P(0,2,4,3,1), P(0,3,5,1); P(4,1,2,5), P(4,0,5)\}\\
P_{4,2} \cup P_{4,2} &\mapsto \{P(0,4,2,5,1), P(0,2,1); P(4,1,3,0,5), P(4,3,5)\}\\
P_{3,3} \cup P_{4,2} &\mapsto \{P(0,2,4,1), P(0,3,5,1); P(4,3,1,2,5), P(4,0,5)\}\\
P_{3,3} \cup P_{3,3} &\mapsto \{P(0,4,3,1), P(0,3,5,1); P(4,1,2,5), P(4,2,0,5)\}
\end{aligned}$$

Table 1: Packings used in Lemma 4.

(A)	(B)	(C)	Conditions
L_m	$S_{4,1,2,2}.C'_3.C'_{m-12}$	$O.L_{m-12}$	$m \geq 15$
$L_{14} \cup C_{\mathbf{n}}$	$S_{4,1,2,\mathbf{2}}.C'_3.P_{4,\mathbf{n-2}}$	$O.L_{n+2}$	$n \geq 8$
$L_{13} \cup C_{\mathbf{n}}$	$S_{4,1,1,\mathbf{3}}.C'_3.P_{4,\mathbf{n-3}}$	$O.L_{n+1}$	$n \geq 8$
L_{12}	$S_{4,1,2,2}.C'_3$	$O.L_0$	
$L_{11} \cup C_{\mathbf{n}}$	$S_{4,1,2,\mathbf{2}}.C'_{\mathbf{3}}.P_{4,\mathbf{n-5}}$	$O.L_{n-1}$	$n \geq 8$
$L_{10} \cup C_{\mathbf{n}}$	$S_{4,1,1,\mathbf{3}}.C'_{\mathbf{3}}.P_{4,\mathbf{n-6}}$	$O.L_{n-2}$	$n \geq 8$
$L_m \cup C_{\mathbf{n}}$	$L_m \cup L_{\mathbf{12-m}}.C'_{\mathbf{n+m-12}}$	$O.L_{m+n-12}$	$3 \leq m \leq 9, m+n \geq 15$
$L_m \cup C_{\mathbf{n}}$	$L_m \cup L_{\mathbf{n}}$	$O.L_0$	$3 \leq m \leq 9, m+n = 12$
$L_6 \cup C_{\mathbf{8}} \cup C_n$	$L_6 \cup P_{\mathbf{4},2}.P_{\mathbf{4},n-2}$	$O.L_{n+2}$	$n \geq 8$
$L_5 \cup C_{\mathbf{9}} \cup C_n$	$L_5 \cup C'_{\mathbf{3}}.P_{\mathbf{2},2}.P_{\mathbf{4},n-2}$	$O.L_{n+2}$	$n \geq 8$
$L_5 \cup C_{\mathbf{8}} \cup C_n$	$L_5 \cup P_{\mathbf{4},3}.P_{\mathbf{4},n-3}$	$O.L_{n+1}$	$n \geq 8$
$L_4 \cup C_{\mathbf{10}} \cup C_n$	$L_4 \cup P_{\mathbf{2},\mathbf{2},\mathbf{2},2}.P_{\mathbf{4},n-2}$	$O.L_{n+2}$	$n \geq 8$
$L_4 \cup C_{\mathbf{9}} \cup C_n$	$L_4 \cup C'_{\mathbf{3}}.P_{\mathbf{2},3}.P_{\mathbf{4},n-3}$	$O.L_{n+1}$	$n \geq 8$
$L_3 \cup C_{\mathbf{11}} \cup C_n$	$L_3 \cup C'_{\mathbf{3}}.P_{\mathbf{4},2}.P_{\mathbf{4},n-2}$	$O.L_{n+2}$	$n \geq 8$
$L_3 \cup C_{\mathbf{10}} \cup C_n$	$L_3 \cup C'_{\mathbf{3}}.P_{\mathbf{3},3}.P_{\mathbf{4},n-3}$	$O.L_{n+1}$	$n \geq 8$
$L_3 \cup C_{\mathbf{8}} \cup C_n$	$L_3 \cup C'_3.P_{\mathbf{4},2}.P_{\mathbf{4},n-5}$	$O.L_{n-1}$	$n \geq 8$

In each case the union of the paths on the right is O and the initial and final links of the left hand expressions (defined before Lemma 2) are mapped to the initial and final links of O. In most cases, the decomposition into paths is a minor variant of a preceding one, so can be checked easily. Note that a packing of $L_4 \cup L_8$ follows from a packing of $L_8 \cup L_4$, and similarly in other cases. Also note that whenever L_6 was used, both versions have been checked. □

Lemma 4 *Suppose $m + \sum m_i \geq 15$ or $m + \sum m_i = 12$ with $m \neq 1, 2$, $m_i \geq 8$. For some subset S and some m' we can pack $L_m \cup (\cup_{i \in S} C_{m_i})$ into $O.L_{m'}$ exactly with initial link matching.*

Proof. The packings shown in Table 1 are available. In each case we can pack graph (A) into (B) by linking up suitable paths, with the link of L_n identified with the initial link. The cycles in bold in (A) pack into the paths and cycles in bold in (B). We can then pack (B) into (C) by Lemma 3. It is easy to check that if $m > 0$ then we must have a subset of one of the forms shown. If $m = 0$, pack some $C_{m_{i_0}}$ into $L_{m_{i_0}}$ first and then use the result with $m > 0$. □

Theorem 5 *If $\sum_{i=1}^{t} m_i \leq \frac{1}{2}(n-7)(n-9) - 164$ then we can pack $\cup_{i=1}^{t} C_{m_i}$ into K_n.*

Proof. By reducing n by at most 7, we can assume that $n \equiv 2$ or 6 mod 12 and $\sum m_i \leq \frac{n}{2}(n-2) - 164$. By Lemma 2 we have a packing of $O^{\cdot a}$ into K_n with $12a \geq \frac{n}{2}(n-2) - 12$ and so $\sum m_i \leq 12a - 152$. We will now show that whenever $\sum m_i \leq 12a - 152$, we have a packing of $\cup C_{m_i}$ into $O^{\cdot a}$.
We can pack four C_3's or two C_6's or $C_6 \cup C_3 \cup C_3$ into O (using $L_3.C'_3 \cup C_3.L_3 \mapsto O$ from Lemma 3), and three C_4's into O (using $P_{2,2,2,2} \cup P_{2,2} \mapsto O$). Therefore by adding cycles of total length at

most $3 \times 3 + 2 \times 4 = 17$, we can pack all C_n with $n = 3, 4, 6$ into some octahedra. We can also pack C_5's and C_7's as follows:

$$(C_5)^{\cup 6} \mapsto P_\emptyset \cup P_{3,3,3,3}.P_{2,2,2,2} \cup P_{2,2}.P_{3,3} \mapsto O.O.L_6$$
$$(C_7)^{\cup 6} \mapsto L_7 \cup P_{3,2}.P_{4,5} \cup L_3.L_4 \cup P_{4,4}.P_{3,3} \mapsto O.O.O.L_6$$

Since we can pack $L_6 \cup L_6$ as O, we see that twelve C_5's pack exactly into $O^{\cdot 5}$, and twelve C_7's pack exactly into $O^{\cdot 7}$. If we don't have twelve C_5's or C_7's, then we add cycles of total length at most $11 \times 5 + 11 \times 7 = 132$ to pack all the C_5's and C_7's into trails of octahedra.

Packing all cycles of length less than 8 into some initial segment of $O^{\cdot a} = O.O \dots O$ and removing this segment, we can now assume that $\sum m_i \le 12a - 152 + 17 + 132 = 12a - 3$ and all the $m_i \ge 8$. Let $m = 12a - \sum m_i \ge 3$ and pack L_m and all the C_{m_i} into $O^{\cdot a}$ inductively using Lemma 4. Either $m + \sum m_i = 12a$ is 12 or at least 15 and we can pack L_m and some cycles into $O.L_{m'}$ by Lemma 4. The sum of m' and the remaining m_i is $12(a-1)$, so we can pack $L_{m'}$ and the remaining cycles into $O^{\cdot a-1}$ by induction. The initial link of $L_{m'}$ is packed into the initial link of $O^{\cdot a-1}$. We therefore have a packing of L_m and all the cycles into $O.O^{\cdot a-1} = O^{\cdot a}$ with the initial link of L_m packed into the initial link of $O^{\cdot a}$.

Discarding L_m and any added cycles from the final packing gives the result. □

Since $\chi'_s(G)$ is just the minimum value of n for which we can pack $L(G)$ into K_n, the following result is immediate.

Corollary 6 *Let G be a 2-regular graph of order n. Then $\chi'_s(G) \le \sqrt{2n} + 24$.*

If $C(G)$ is the minimum number of colors needed in a *not necessarily proper* vertex-distinguishing edge-coloring of G then for 2-regular graphs

$$\sqrt{2n} - \frac{1}{2} \le C(G) \le \chi'_s(G) \le \sqrt{2n} + 24.$$

Hence we have determined both $C(G)$ and $\chi'_s(G)$ up to the addition of a constant.

The methods above can be refined to prove the following much stronger result (see [2]).

Theorem 7 (Corollary 2 of [2]) *Let $L = \sum_{i=1}^{t} m_i$ with $m_i \ge 3$. Then we can write some subgraph of K_n as an edge disjoint union of circuits of length $m_1, \dots, m_t$ if and only if either*

1. *n is odd, $L = \binom{n}{2}$ or $L \le \binom{n}{2} - 3$, or*
2. *n is even, $L \le \binom{n}{2} - \frac{n}{2}$.*

This stronger result gives the exact values of $\chi'_s(G)$ for all 2-regular G, and in particular implies the Burris and Schelp conjecture for these graphs.

Corollary 8 *Let G be a vertex-disjoint union of cycles, and let $n_2(G) = |V(G)| \le \binom{k}{2}$, with k chosen as small as possible. Then $\chi'_s(G) = k$ or $k + 1$.*

The proof of Theorem 7 is much longer and more technical than Theorem 5, but is based on the same ideas as the proof of Theorem 5. Note that Corollary 6 only differs from Corollary 8 in the additive constant in the bound.

3 Unions of Paths

In this section we prove the conjecture in the case when G is a vertex-disjoint union of p paths $P_{l_1+1} \cup \ldots \cup P_{l_p+1}$ where $l_i \geq 1$. As described in the introduction, this is equivalent to packing the line graph $L(G) = P_{l_1} \cup P_{l_2} \cup \ldots \cup P_{l_p}$ into K_n with the $2p$ endpoints of the paths mapped to distinct vertices in K_n. Note in particular that we must have $n \geq 2p$. Write $L = \sum_{i=1}^{p} l_i$ and note that $n_1(G) = 2p$ and $n_2(G) = L$.

Theorem 9 *The following conditions are both necessary and sufficient for packing $\cup_{i=1}^{p} P_{l_i}$ into K_n with endpoints mapped to distinct vertices:*

$$\begin{array}{ll} L = \binom{n}{2} \text{ or } L \leq \binom{n}{2} - 3 & \text{if } r = 0, \\ L \leq \binom{n}{2} - \frac{r}{2} & \text{if } r > 0 \text{ and } r \text{ (or } n\text{) is even,} \\ L \leq \binom{n}{2} - p & \text{if } r \text{ (or } n\text{) is odd.} \end{array}$$

where $n = 2p + r$ and $L = \sum_{i=1}^{t} l_i$. In particular, $L \leq \binom{n-1}{2}$ is always sufficient.

Proof.
1. Proof that the conditions are necessary.
Consider the image G of the packing in K_n. The degrees of $2p$ of the vertices must be odd and the remaining r vertices must have even degree. Now consider the edge complement G^c of G in K_n. If r (and hence n) is odd, G^c will have $2p$ odd degree vertices. Hence G^c will have at least p edges. If r is even then G^c will have r odd degree vertices and at least $r/2$ edges. If $r = 0$ then every vertex of G^c has even degree and so G^c has either no edges or at least three edges.

2. Proof that the conditions are sufficient.
Order the paths P_{l_i} so that $l_1 \geq l_2 \geq \ldots \geq l_p$. Since K_n contains a set of $\lfloor \frac{n}{2} \rfloor \geq p$ independent edges, we are done in the case when all $l_i = 1$, so we may assume $l_1 \geq 2$.

Now consider the case $p = 1$. If n is odd and $l_1 = \binom{n}{2} - 1$, remove one edge from an Eulerian circuit of K_n. Otherwise, if $l_1 \geq 4$ we can pack a cycle of length $l_1 - 1$ into some subgraph G of K_n using Theorem 7 ($l_1 - 1 \leq \binom{n}{2} - \frac{n}{2}$ if n even and $l_1 - 1 \leq \binom{n}{2} - 3$ if n odd). Since K_n is connected, there must be some unused edge $uv \in E(G^c)$ with u meeting G. Adding this edge to the circuit gives a trail of length l_1 with distinct endpoints as required. If $l_1 \leq 3$ the result is trivial.

Now assume $p \geq 2$, $l_1 \geq 2$, so $n \geq 4$. Let $\lambda = l_1 + l_2 - 2$, so that $\lambda \geq l_2$. We shall try to pack paths of lengths $\lambda, l_3, \ldots$ into K_{n-2} by induction. This may fail due to the fact that the total length is too large, so we will reduce the lengths. If $\lambda \geq 4$ and n is even, reduce λ by three. Now reduce each λ or l_i, $i \geq 3$ by multiples of four until we have removed a total length of $2n - 5$ (n even), or $2n - 6$ (n odd) or until we have reduced all the lengths to at most four, (if $\lambda < 4$ then $l_i < 4$ for all $i \geq 3$). Call these reduced lengths $\lambda', l'_3, \ldots, l'_p$ and pack trails of these lengths into K_{n-2}. We will show that this will succeed in almost all cases.

If we have removed a total of $2n - 5$ or $2n - 6$ from the lengths, the total reduced length L' will be at most $L - (2n - 3)$ (n even) or $L - (2n - 4)$ (n odd). If $L = \binom{n}{2} - \delta$ then $L' = \binom{n-2}{2} - \delta'$ where $\delta' = \delta$ when n even and $\delta' = \delta - 1$ when n is odd. Since p has been reduced by one and r is the same, this L' satisfies the conditions for $n - 2$ and we can pack the paths by induction. If we cannot reduce the path lengths this much, the remaining paths must all be of length at most four. In this case $L' \leq 4(p - 1) \leq 4\lfloor \frac{n-2}{2} \rfloor$ which also satisfies the conditions when $n \geq 7$.

The cases when $n \leq 6$ must be verified by a case by case analysis. In fact, the above algorithm works except in some cases when $n = 6$, $p = 3$ and $l_1 + l_2 \leq 9$ and in some cases when $n = 5$, $p = 2$ and $l_1 + l_2 \leq 6$. For each of these cases the theorem can be checked directly.

We now add back the two remaining vertices a and b of K_n and construct trails of the original lengths. Let the trail of length λ' go from vertex u to v in K_{n-2}. Let u' be any vertex on this trail which is a distance at most $l_1 - 1$ along the trail from u and distance at most $l_2 - 1$ from v and so that the distance from v is equivalent to $l_2 - 1$ mod 2. Such a vertex exists since $\lambda' \leq (l_1 - 1) + (l_2 - 1)$ and $\lambda' \geq 1$. For each of the P_{l_i}, $i \geq 3$ that have been shortened, pick an endvertex v_i of the trail of length l_i' in K_{n-2} not equal to u'. Pick $(l_i - l_i' - 2)/2$ paths of length two of the form $P(a, x, b)$ where x is not any v_i or u'. Linking up these paths (there are an odd number of them) together with the edges $v_i a$ and $v_i b$ and the trail of length l_i' gives a trail of length l_i in K_n with the same endvertices as the trail of length l_i' in K_{n-2}. We now construct trails corresponding to P_{l_1} and P_{l_2}. Construct a trail from v to u' (using part of the trail of length λ') to a (via the edge $u'a$) and then some number of paths of length two between a and b until we have a trail of length l_2 from v to either a or b. The remaining trail of length l_1 can be made up from the other part of the trail of length λ' from u to u' to b (via $u'b$) and then using the remaining trails of length one or two between a and b. (The edge ab is used if n is even and $\lambda \geq 4$). The resulting trail of length l_1 goes from u to either b or a (distinct from the endpoints of the trail of length l_2). Since the original paths were shortened by at most $2n - 5$, we do not run out of paths $P(a, x, b)$ of length two from a to b. □

As a consequence, the exact value of $\chi_s'(G)$ when G is a union of paths is just the smallest n satisfying the conditions of Theorem 9 where $L = n_2(G)$ and $2p = n_1(G)$. In particular, Conjecture 1 now follows when G is a union of paths.

Corollary 10 *Let G be the vertex-disjoint union of paths with each path of length at least two. Let $n_1(G) \leq k$ and $n_2(G) \leq \binom{k}{2}$, with k chosen as small as possible. Then $\chi_s'(G) = k$ or $k + 1$.*

It is worth noting that in both the cases when G is a union of cycles and when G is a union of paths, the cases when $\chi_s'(G) = k + 1$ occur only when forced by parity considerations.

4 Unions of Cycles and Paths

In this section we shall consider the case when G is a general vdec-graph with $\Delta(G) = 2$. Such a G is a vertex-disjoint union of paths P_{l_i+1}, $i = 1, \ldots, p$ (of length at least two), cycles C_{m_i}, $i = 1, \ldots, t$ and possibly a single isolated vertex. As before, we translate the problem into a packing problem on the line graph $L(G)$. In this case, we need to pack both paths P_{l_i} (of lengths one less than those of G) and cycles C_{m_i} into K_n with the endpoints of the paths mapped to $2p$ distinct vertices in K_n. In terms of the original graph G, $n_1(G) = 2p$ and $n_2(G) = L = \sum_{i=1}^{t} m_i + \sum_{i=1}^{p} l_i$.

For such general graphs we do not have an exact result. However we will show that if $n_1 \leq k$ and $n_2 \leq \binom{k}{2}$ then we can strongly color G with at most $k + 5$ colors.

Lemma 11 *If all but at most one of the paths P_{l_i} has length one or two, $n \geq p$ and the total length L of all paths and cycles is at most $2n(n - 1) - 3$ then we can pack the paths and cycles into some subgraph of K_{2n+1} with the endpoints of the paths mapped to $2p$ distinct vertices.*

Proof. Let l_1 be the length of a longest path. Let $m = l_1 - 1$ if $l_1 \geq 4$, $m = 1$ if $l_1 = 3$ and $m = 0$ otherwise (including the case when there are no paths). Add an additional cycle so that the total length of all cycles is exactly $2n(n-1) - m$. This cycle will have length at least $3 + l_1 - m \geq 3$. By Theorem 7, we can pack these cycles (and an additional C_m when $m > 1$) into K_{2n} with n edges remaining ($n + 1$ edges if $m = 1$). These missing edges must form a 1-factor of K_{2n} (or a $K_{1,3}$ and a set of independent edges if $m = 1$). If $m = 1$, add a path of length $l_1 = 3$ by taking a path of length two in $K_{1,3}$ and adding an edge to the $2n+1$'th vertex a. If $m > 1$ pick an edge of the missing 1-factor which meets C_m in the packing of K_{2n} and combine to get a trail of length $m + 1 = l_1$. We now match up the remaining paths with the remaining unused independent edges of K_{2n}. To pack a path of length one, just use the corresponding edge uv. For paths of length two use $P(u, a, v)$. We now have a packing as desired. □

Note, that if we only use the weaker result for cycle packing given in Theorem 5, the proof still holds (with some minor modifications) if the total length L is at most $2(n-3)(n-4) - 167$. (The construction in Theorem 5 leaves out a 1-factor in K_{2n} and if $l_1 = 3$ pack P_{l_1} in the missing O of Lemma 2.)

Theorem 12 *If $n + 1 \geq p$ and $L \leq 2n(n-1) - 3$, then we can pack all the paths and cycles into K_{2n+3} with the endpoints of the paths mapped to $2p$ distinct vertices.*

Proof. We use a similar strategy to the case when we have only paths. We may assume the paths are of lengths l_i with $l_1 \geq l_2 \geq \cdots \geq l_p$. If $p < 2$ or $l_2 < 3$ then we are done by Lemma 11, so assume $l_1, l_2 \geq 3$. Now let $\lambda = l_1 + l_2 - 2$ and consider packing the cycles and paths of lengths $\lambda, l_3, \ldots, l_p$ into K_{2n+1}. As before, the total length may be too large, so we shorten the paths by multiples of two with the restriction that the total reduction in length must be a multiple of four and be at most $4n$. In other words, write $\lambda' = \lambda - 2k_2$ and $l'_i = l_i - 2k_i$, $i \geq 3$ with $\lambda', l'_i \geq 1$, $\sum k_i$ even, and $\sum 2k_i \leq 4n$. If we run out of paths to shorten, at most one path can have length more than two (and even this path has length at most four). We can therefore pack the paths and cycles into K_{2n+1} by the previous lemma. Otherwise the total length of paths and cycles is now at most $2n(n-1) - 3 - 4n - 2 \leq 2(n-1)(n-2) - 3$ so we can pack them into K_{2n+1} by induction on n.

We now put back the two remaining vertices a and b and construct trails of the correct lengths. Let the trail of length λ' go from vertex u to v in K_{2n+1}. Let u' be any vertex on the trail from u to v which is a distance at most $l_1 - 1$ along the trail from u and distance at most $l_2 - 1$ from v and so that these distances are equivalent to $l_1 - 1$ or $l_2 - 1$ mod 2. For each of the P_{l_i}, $i \geq 3$ that have been shortened, pick an endvertex v_i of the path in K_{2n+1} not equal to u'. Connect v_i to a and b and then add in $k_i - 1$ paths of length two of the form $P(a, x, b)$ where x is not any v_i or u'. If an even multiple of two has been removed from the length, we can link up the paths to give a path of length l_i as before. If an odd multiple of two has been removed, pick another path l_j which also has had an odd multiple of two removed. We now let P_{l_i} go along the ith trail in K_{2n+1} to v_i, then to a, along an even number of length two paths from a to b and then from a to v_j. We let P_{l_j} go along the jth trail in K_{2n+1} to v_j, then to b, along an even number of length two paths from a to b and then from b to v_i. We now have trails of lengths l_i and l_j with distinct endpoints. Two of the endpoints have been swapped, however this is of no importance.

The path P_{l_1} is packed into the trail from u to u' to a and some number of paths of length two between a and b. The result is a trail of length l_1 from u to either a or b. The path P_{l_2} is packed along the trail in K_{2n+1} from v to u', then to b, along some paths of length two between a and b. If we reduced λ by an even multiple of two, then we are done as before.

Now assume we reduced λ by an odd multiple of two so we have one other unpacked path P_{l_j} with $j \geq 3$ and $l_j \geq 3$. Since $\lambda' < \lambda$, either the distance from u' to u along the trail of length λ' is less than $l_1 - 1$, or the distance from u' to v is less than $l_2 - 1$. We can assume without loss of generality that the first case holds and the distance from u to u' is at most $l_1 - 3$.

The trail of length l_j will go along ua, an even number of paths of the form $P(a, x, b)$, then av_j and along the trail of length l'_j in K_{2n+1}. The trail of length l_2 will go along the trail from v to u', along $u'b$, and then along some paths $P(a, x, b)$ to either a or b. If $u' \neq u$ then the trail of length l_1 will go from v_j, along v_jb, along bu, then along the trail to u', along $u'a$ and some paths of the form $P(a, x, b)$ ending at either b or a. If $u' = u$ then the trail of length l_1 will just go from v_j along v_jb and some paths of the form $P(a, x, b)$. □

Corollary 13 *Let G be any vdec-graph with $\Delta(G) = 2$. Let $n_1(G) \leq k$ and $n_2(G) \leq \binom{k}{2}$, with k chosen as small as possible. Then $k \leq \chi'_s(G) \leq k + 5$.*

Proof. Set $n = \lceil \frac{k+1}{2} \rceil$ in Theorem 12. Then $n + 1 \geq \frac{k}{2} \geq \frac{n_1}{2} = p$ and $2n(n - 1) - 3 \geq \frac{1}{2}(k + 1)(k - 1) - 3 \geq \binom{k}{2} \geq n_2 = L$ for $k \geq 6$. The result follows when $k \geq 6$ since $2n + 3 \leq k + 5$. The cases when $k < 6$ can be checked easily. Indeed, there are more colors available than edges when $k < 5$. □

Note that if we use Theorem 5 instead of Theorem 7 throughout, we get the slightly weaker bound $k \leq \chi'_s(G) \leq k + 25$.

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