Bond percolation with attenuation in high dimensional Voronoĭ tilings

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Abstract

Let $\mathcal{P}$ be the set of points in a realization of a uniform Poisson process in $\mathbb{R}^n$. The set $\mathcal{P}$ determines a Voronoĭ tiling of $\mathbb{R}^n$. Construct an infinite graph $\mathcal{G}$ with vertex set $\mathcal{P}$ and edges joining vertices when the corresponding Voronoĭ cells share a $(n-1)$-dimensional boundary face. We consider bond percolation models on $\mathcal{G}$ obtained by declaring each edge xy of $\mathcal{G}$ open independently with probability $p(\|x-y\|)$, depending only on the Euclidean distance $\|x-y\|$ between the vertices. We give some sufficient conditions on $p(t)$ that ensures that an infinite connected component (i.e., percolation) occurs, or does not occur. In particular, we show that for $p(t)=p$ is a constant, there is a phase transition at a critical probability $p=p_c(n)$, where $2^{-n}(5n\log n)^{-1} \le p_c(n) \le C2^{-n}\sqrt{n}\log n$. We also show that if $p(t)=e^{-\lambda t}$ then there is a phase transition at a critical parameter $\lambda=\lambda_c(n)$, where $\lambda_c(n)=(\log_e 2+o(1))n/2r_n$, where r_n is the radius of the n-dimensional sphere that, on average, contains a single point of $\mathcal{P}$.

1 Introduction

Take a Poisson process on $\mathbb{R}^n$ with constant density. A realization of the process consists (almost surely) of a countably infinite subset $\mathcal{P} \subseteq \mathbb{R}^n$. For a given realization, the points in it will be called *Poisson points*. Corresponding to each Poisson point x, we construct its *Voronoĭ cell* which consists of those points of $\mathbb{R}^n$ that are closer to x than to any other Poisson point. We note that almost surely $\mathcal{P}$ is discrete, so that the Voronoĭ cells are convex open sets. Since $\mathcal{P}$ is countable, it is clear that the points that are equidistant

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from two or more Poisson points have Lebesgue measure 0 so that the Voronoĭ cells form a partition of $\mathbb{R}^n$ up to sets of measure 0, the *Voronoĭ tessellation* associated to $\mathcal{P}$. This random tessellation is said to be a *random Voronoĭ tessellation* of $\mathbb{R}^n$. Random Voronoĭ tessellations have been studied for over a century and a half, first in crystallography and then in mathematics (see §8.3 of [4], and the references therein). In particular, Delesse [5] estimated the volume of a crystal, and Gilbert [6] studied the expectations of the surface area, the number of faces, and other parameters.

It is easily seen that for almost every realization of the Poisson process, the Voronoĭ cells are all of finite volume and all have finitely many $(n-1)$-dimensional faces. Two Voronoĭ cells are said to be neighbors if they share a $(n-1)$-dimensional face. Define a graph $\mathcal{G}$ with vertex set $\mathcal{P}$ and points adjacent iff their Voronoĭ cells are neighbors. Note that almost surely, $\mathcal{G}$ is *locally finite*, i.e., every vertex of $\mathcal{G}$ has finite degree.

Percolation on this graph $\mathcal{G}$ is said to be a *random Voronoĭ percolation* . In 1963, Frisch and Hammersley called for the study of percolation on geometric graphs, including Voronoĭ percolation. This challenge was first taken up by physicists, who performed numerous computer experiments to estimate the critical probabilities (see [8], [9], [13] among many others).

The mathematical study of random Voronoĭ percolation started much later: first, in a series of papers, Vahidi-Asl and Wierman [10, 11, 12] studied first-passage percolation on random Voronoĭ percolation, and then Gravner and Griffeath [7] investigated cellular automata on random colourings of a random Voronoĭ tessellation. Balister, Bollobás and Quas [1] gave bounds on the critical probability of site percolation in high dimensions, and Bollobás and Riordan [2, 3] proved that the critical probability of site percolation in the plane is 1/2, and so is the critical probability of the so-called Johnson–Mehl tessellation in the plane.

Our aim in this paper is to study bond percolation in random Voronoĭ percolation in high dimensions, i.e., bond percolation on the random graph $\mathcal{G}$ in $\mathbb{R}^n$ we defined above. In fact, we shall consider two cases of a rather general bond percolation model. Fix a function $p\colon [0,\infty) \to [0,1]$ and declare an edge xy of $\mathcal{G}$ *open* with probability $p(\|x-y\|)$, independently of the state of all other edges. Here $\|x-y\|$ is the Euclidean (ℓ_2) distance between the points $x, y \in \mathcal{P}$. Write $\mathcal{G}_{\text{open}}$ for the subgraph consisting of the open edges in $\mathcal{G}$. We shall be particularly interested in two cases.

1. Constant $p \in [0,1]$. This corresponds to the usual concept of independent bond percolation on an infinite graph.

2. Exponential attenuation, $p(t) = e^{-\lambda t}$, $\lambda > 0$.

The model is said to *percolate* if $\mathcal{G}_{\text{open}}$ has an infinite component. The existence of an infinite component in $\mathcal{G}_{\text{open}}$ is unaffected by the state of any finite set of edges, so is a tail event. Thus by the Kolmogorov 0-1 law, for any choice of $p(t)$, the probability of percolation is either 0 or 1. It is a common situation when studying percolation that when varying a parameter of the model, there exists a critical value of the parameter where the probability of percolation jumps from 0 to 1. In this case we say the model has a *phase transition* at this value of the parameter.

Define r_n to be the radius of the n-dimensional sphere which contains on average one Poisson point. By a uniform scaling of $\mathbb{R}^n$ we may, and shall, assume that the Poisson process has intensity one, so that r_n is the radius of the n-dimensional sphere with unit volume. In this case $r_n = (n/2)!^{1/n}/\sqrt{\pi} \sim \sqrt{n/2\pi e}$. We shall prove the following two results.

Theorem 1. *If n is sufficiently large and*

$$\int_0^{2^n(4n\log n)} p(t^{1/n}r_n)\,dt < 0.9$$

then there is almost surely no percolation.

Theorem 2. *There exists an absolute constant C such that if $p(t) > C\,2^{-n}\sqrt{n}\,\log n$ for all $t < 2r_n$, then there is almost surely percolation.*

From these we can deduce the following.

Corollary 3. *If $p(t) = p$ is a constant, then a phase transition occurs at $p = p_c(n)$ where for sufficiently large n,*

$$2^{-n}(5n\log n)^{-1} \le p_c(n) \le C2^{-n}\sqrt{n}\log n.$$

Proof. A simple coupling argument shows that if there is percolation at a given value of p then there is percolation at all larger values of p. Thus there is a critical $p_c(n) \in [0,1]$ such that percolation occurs for all $p > p_c(n)$ and does not occur for all $p < p_c(n)$. Theorem 1 shows that there is no percolation when $p < 2^{-n}(5n\log n)^{-1}$ and Theorem 2 shows that there is percolation for $p > C\,2^{-n}\sqrt{n}\,\log n$. The result follows. $\square$

Corollary 4. *If $p(t) = e^{-\lambda t}$, then a phase transition occurs at $\lambda = \lambda_c(n)$ where*

$$\lambda_c(n) = (\log 2 + o(1))n/2r_n.$$

Proof. As above, monotonicity of $p(t)$ as λ varies implies that there is a critical $\lambda_c(n) \in [0,\infty]$. Setting $\lambda = (n/2r_n)\log(2+\varepsilon)$, $\varepsilon > 0$, and $t = x^n$,

$$\begin{aligned}
\int_0^{2^n(4n\log n)} p(t^{1/n}r_n)\,dt &= \int_0^{2+o(1)} nx^{n-1}(2+\varepsilon)^{-nx/2}\,dx \\
&\le \int_0^{2+o(1)} n(x(2+\varepsilon)^{-x/2})^{n-1}\,dx \\
&\le (2+o(1))nc^{n-1},
\end{aligned}$$

where $c = \sup_{x \in [0,2+o(1)]} x(2+\varepsilon)^{-x/2} = 2/(2+\varepsilon) + o(1) < 1$ (for $2+\varepsilon < e$). Thus for large n percolation almost surely does not occur by Theorem 1.

Setting $\lambda = (n/2r_n)\log(2-\varepsilon)$, $\varepsilon > 0$, gives that for $t \leq 2r_n$, $p(t) \geq (2-\varepsilon)^{-n} > C\,2^{-n}\sqrt{n}\,\log n$ for large n. Thus by Theorem 2 percolation almost surely does occur.

Thus for any $\varepsilon > 0$, $(n/2r_n)\log(2-\varepsilon) \leq \lambda_c(n) \leq (n/2r_n)\log(2+\varepsilon)$ for sufficiently large n. The result follows. $\square$

2 The Lower Bound

We follow the proof of the lower bound for site percolation given in [1]. Starting from a Voronoĭ percolation process on $\mathbb{R}^n$, we introduce a site percolation process on $\mathbb{Z}^n$, which we use to prove the absence of percolation in the original Voronoĭ process. The $\mathbb{Z}^n$ process fails to have the usual independence between sites, but has instead a dependence that is of finite range.

To construct the site percolation process on $\mathbb{Z}^n$, we start off with a realization of the Voronoĭ process and divide $\mathbb{R}^n$ into cubes of side length R, where $R = n^{1.1}r_n$. Note that $R = \Theta(n^{1.6})$. Each cube corresponds to a vertex in $\mathbb{Z}^n$ and two such cubes are *adjacent* if their closures intersect, which is equivalent to sites being diagonally adjacent (ℓ_∞-distance 1 apart) in $\mathbb{Z}^n$. For a set C, $B_r(C)$ will denote $\{x : \inf_{y \in C} \|x-y\| < r\}$. We will write $B_r(x)$ for $B_r(\{x\})$.

A cube C in $\mathbb{R}^n$ (or the corresponding vertex in $\mathbb{Z}^n$) is said to be *open* if either

(1) there exists a point $x \in B_{R/4}(C)$ such that $B_{R/4}(x)$ contains no point of the underlying Poisson process; or

(2) there exists a curve γ in $B_{R/4}(C)$ such that $\gamma(0) \in \partial C$, $\gamma(1) \in \partial B_{R/4}(C)$, and for all $t \in [0,1]$, $\gamma(t)$ either lies in the interior of a Voronoĭ cell, or in the $(n-1)$-dimensional interior a face joining two cells that are adjacent in $\mathcal{G}_{\text{open}}$.

Lemma 5. *The openness of a cube C is determined by the restriction of the the Poisson process to $B_{R/2}(C)$ together with the openness of any edge joining Poisson points in $B_{R/2}(C)$.*

Proof. The cube C is open if (1) holds or if (2′) holds, where (2′) is the condition that (1) fails but (2) holds.

Clearly whether or not condition (1) is satisfied is determined by the restriction of the Poisson process to $B_{R/2}(C)$. It is sufficient to show that (2′) is determined by the restriction of the Poisson process to $B_{R/2}(C)$. Given that (1) fails, every point of $B_{R/4}(C)$ is within $R/4$ of a point of the Poisson process. The Voronoĭ cells restricted to $B_{R/4}(C)$ are therefore determined by the restriction of the Poisson process to $B_{R/2}(C)$. The information in the subgraph of $\mathcal{G}_{\text{open}}$ induced by the Poisson points in $B_{R/2}(C)$ is thus sufficient to determine whether or not (2′) holds. $\square$

It follows from Lemma 5 that openness of two sites in the $\mathbb{Z}^n$ process are independent provided that they are not (diagonally) adjacent.

Lemma 6. *The probability of percolation in the $\mathbb{Z}^n$ process defined above is bounded below by the probability of percolation in the Voronoĭ process.*

Proof. If $\mathcal{G}_{\text{open}}$ contains an infinite component, then as $\mathcal{G}$ is locally finite and by compactness, there must be an infinite path in $\mathcal{G}_{\text{open}}$. Thus there is an unbounded curve γ in $\mathbb{R}^n$ such that for all t, $\gamma(t)$ either lies in the interior of a Voronoĭ cell, or in the $(n-1)$-dimensional interior of a face joining two cells that are adjacent in $\mathcal{G}_{\text{open}}$. For any cube C that intersects γ, one can take a subpath of γ satisfying (2) above, so C is open in the $\mathbb{Z}^n$ process. The set of all cubes C meeting γ then gives an infinite open component in the $\mathbb{Z}^n$ process. $\square$

Hence to show that the Voronoĭ process does not percolate, it is sufficient to show that the site process does not percolate. Before proving this, we state three lemmas from [1].

Lemma 7 (Lemma 4 of [1])**.** *Let p be the probability that a given cube is open. If $p < 9^{-n}$ then there is almost surely no percolation in the $\mathbb{Z}^n$ process.* $\square$

Lemma 8 (Lemma 5 of [1])**.** *Let $R = r_n n^{1.1}$ and write A_1 for the event that each point in $B_{R/2}(C)$ has a Poisson point within distance $r_n(4n \log n)^{1/n}$. Then $\mathbb{P}(A_1^c) = o(9^{-n})$.* $\square$

Lemma 9 (Lemma 6 of [1])**.** *Let $R = r_n n^{1.1}$ and write A_3 for the event that there are at most $2(2R)^n$ Voronoĭ cells intersecting ∂C. Then $\mathbb{P}(A_3^c) = o(9^{-n})$.* $\square$

Lemma 10. *Let A be the event that the cube C of side $R = r_n n^{1.1}$ is open. Then under the assumptions of Theorem 1, $\mathbb{P}(A) = o(9^{-n})$.*

Proof. By the previous two lemmas, we may restrict our attention to configurations belonging to $A_1 \cap A_3$. Since $R/4 > r_n(4n \log n)^{1/n}$ for large n, condition (1) fails automatically. We note that by A_1, any pair of Poisson points in $B_{R/2}(C)$ whose Voronoĭ cells meet within $B_{R/4}(C)$ must be within distance $2r_n(4n \log n)^{1/n}$ of each other, otherwise the point where the cells meet would contradict A_1. Thus the path γ in (2) must pass through cells corresponding to Poisson points within $2r_n(4n \log n)^{1/n}$ of each other. Thus for condition (2) to hold, there needs to be a path of open Voronoĭ cells consisting of at least $(R/4)/(2r_n(4n \log n)^{1/n})$ cells. But for sufficiently large n, $(R/4)/(2r_n(4n \log n)^{1/n}) > n^{1.1}/10$. It follows that $\mathbb{P}(A \cap A_1 \cap A_3)$ is bounded above by the probability that there exists an open path of tiles of length $n^{1.1}/10$ starting from ∂C.

Write $\mathcal{G}'$ for the graph with the same vertex set as $\mathcal{G}$, but vertices joined when they are within distance $2r_n(4n \log n)^{1/n}$. Let $\mathcal{G}'_{\text{open}}$ be defined similarly. (We couple $\mathcal{G}_{\text{open}}$ and $\mathcal{G}'_{\text{open}}$ so that when an edge exists in both then they are either both open or both closed.)

Then by the above, $\mathbb{P}(A \cap A_1 \cap A_3)$ is bounded above by the probability that there exists an open path of tiles in $\mathcal{G}'_{\text{open}}$ of length $n^{1.1}/10$ starting from ∂C. Write

$$E = \int_0^{2^n(4n\log n)} p(t^{1/n} r_n)\, dt.$$

Then E is the expected number of neighbours in $\mathcal{G}'_{\text{open}}$ of a vertex. (The integration is over volume in consecutive spherical shells of radius $t^{1/n} r_n$ about the vertex. Such a shell encloses a volume of t.) If we condition on the existence of a path $P = v_1 \dots v_k$ in $\mathcal{G}'_{\text{open}}$, then the expected number of neighbours of v_k other than the v_i, $i < k$, is exactly E. This is because (unlike in $\mathcal{G}_{\text{open}}$) the existence of $v_1, \dots, v_{k-1}$ does not affect whether or not any other Poisson point is a neighbour of v_k, and conditioned on the existence of $v_1, \dots, v_k$ the remaining Poisson points are still distributed according to a Poisson process. Thus if we write E_k for the expected number of paths of edge length k in $\mathcal{G}'_{\text{open}}$ starting at a fixed vertex v_1, then $E_0 = 1$ and $E_{k+1} = E_k E$ for all $k \geq 0$. Hence $E_k = E^k$.

Writing E_o for the expected value of the number of paths of length $n^{1.1}/10$ in $\mathcal{G}_{\text{open}}$ starting from ∂C times the indicator function of $A_1 \cap A_3$,

$$E_o \leq 2(2R)^n E_{n^{1.1}/10} = 2(2R)^n E^{n^{1.1}/10} = O(n^{2n} 0.9^{n^{1.1}/10}) = o(9^{-n}).$$

Since the expected value is an upper bound for the probability of existence, we see that $\mathbb{P}(A \cap A_1 \cap A_3) = o(9^{-n})$ and hence $\mathbb{P}(A) = o(9^{-n})$ as required. □

Theorem 1 now follows from Lemmas 6, 7, and 10.

3 The Upper Bound

We shall bound the probability of the Voronoĭ process percolating by comparing it with site percolation on the cells. For this we use the following result from [1].

Theorem 11. *There exists a constant C such that for $p > C\, 2^{-n}\sqrt{n} \log n$ the site percolation on the Voronoĭ cells obtained by declaring each cell independently open with probability p, has infinite clusters with probability 1. Moreover, this result remains true if we delete edges of length more than $2\eta r_n$ where $\eta^n = (1 - p_c)/(32 r_n)$ and p_c is the critical probability for oriented site percolation in $\mathbb{Z}^2$.*

Proof of Theorem 2. We note that $\eta < 1$. Let $\mathcal{G}'$ be $\mathcal{G}$ with all edges that are longer than $2r_n$ deleted. Then Theorem 11 implies that we have site percolation on $\mathcal{G}'$ when vertices are open with probability $p > C\, 2^{-n}\sqrt{n} \log n$. However, in general, site percolation on any graph with a given value of p implies bond percolation on the same graph with the same p (see [4]). Thus we have bond percolation on $\mathcal{G}'$ with this value of p, and hence we have an infinite component in $\mathcal{G}_{\text{open}}$ provided $p(t) \geq p$ for $t \leq 2r_n$. □

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