THREE RESULTS IN RECURRENCE

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Proofs of some recent results of Kriz and Forrest are presented which distinguish between four different kinds of "sets of recurrence" arising naturally in connection with topological and measure-preserving dynamical systems.

0. Suppose that $(X, \Psi, \mu, T)$ is an invertible measure preserving system, that is, $(X, \Psi, \mu)$ is a measure space with $\mu(X) = 1$ and $T : X \to X$ is a bijection mod 0, with $\mu(A) = \mu(TA) = \mu(T^{-1}A)$ for all measurable sets $A$. (By bijection mod 0 we mean that there exists $X' \in \Psi$, $\mu(X') = 1$, such that the restriction of $T$ to $X'$ is a bijection.) Then

**Theorem 0.1.** (Poincaré) If $A \in \Psi$ with $\mu(A) > 0$, then for some natural number $n$, $\mu(A \cap T^n A) > 0$.

This is the classical Poincaré recurrence theorem. A refinement is

**Theorem 0.2.** If $E$ is an infinite subset of the natural numbers and $\mu(A) > 0$, then for some $k$ in the set

$$E - E = \{ n - m : n > m, n, m \in E \}$$

we have $\mu(A \cap T^k A) > 0$.

**Proof.** The sets $T^n A, n \in E$, being of equal positive measure, cannot be pairwise disjoint in the finite measure space $X$, hence for some $n, m \in E$, $n > m$, $\mu(T^n A \cap T^m A) > 0$. Since $T$ is measure preserving we have $\mu(A \cap T^n - m A) > 0$.

Theorem 0.2 motivates the following definition.

**Definition 0.3.** Suppose $E \subset \mathbb{N}$ is the set of natural numbers. $E$ will be called a set of measure theoretic recurrence if for any invertible measure preserving $(X, \Psi, \mu, T)$ and $A$, $\mu(A) > 0$, there exists $n \in E$ such that $\mu(A \cap T^n A) > 0$.

Hence the set $E - E$ of Theorem 0.2 is a set of measure theoretic recurrence. Our concern here is not really invertible measure preserving systems but subsets of $\mathbb{N}$.

**Definition 0.4.** Suppose that $B \subset \mathbb{N}$. The upper density of $B$ is the number

$$\overline{d}(B) = \limsup_{N \to \infty} \frac{\#(\{1, 2, \ldots, N\} \cap B)}{N}$$

A subset $E \subset \mathbb{N}$ is called density intersective if whenever $\overline{d}(B) > 0$ we have $\overline{d}(B \cap (B + n)) > 0$ for some $n \in E$. 

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**Proposition 0.5.** $E$ is a set of measure theoretic recurrence if and only if $E$ is density intersecive.

The proof of Proposition 0.5 is implicit in [Furstenberg, 1981] and explicit in [Bergelson, 1987]. Suppose now that instead of an invertible measure preserving system, we have a minimal topological dynamical system $(X, T)$. Here $X$ is a compact metric space and $T : X \to X$ is a homeomorphism such that for every non-empty proper closed subset $C \subset X$, $C \neq T(C)$.

**Definition 0.6.** A subset $E$ of $\mathbb{N}$ is said to be a set of topological recurrence if whenever $(X, T)$ is a minimal topological dynamical system, and $U$ is an open subset of $X$, there exists $n \in E$ such that $U \cap T^n U \neq \emptyset$.

**Proposition 0.7.** $E$ is a set of topological recurrence if and only if for any minimal system $(X, T)$, the set of points $x \in X$ with the property that whenever $U$ is open and contains $x$, there exists $r \in E$ such that $T^r x \in U$, is residual in $X$.

**Proof.** Set

$$A_n = \{x \in X : \text{there exists } r \in E \text{ such that } d(x, T^r x) < \frac{1}{n}\}$$

Then each $A_n$ is open and dense so that $\bigcap_{n=1}^\infty A_n$ is the required residual set. The converse is trivial since residual sets are necessarily dense.

**Definition 0.8.** By an $r$-coloring of $\mathbb{N}$ we mean a partition $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$. The indices $i$, $1 \leq i \leq r$, are the colors. A subset $E \subset \mathbb{N}$ is said to be $r$-intersecive if for every $r$-coloring of $\mathbb{N}$ there exists a color $i$ such that $(C_i - C_i) \cap E$ is non-empty. $E$ is said to be chromatically intersecive provided it is $r$-intersecive for all $r \in \mathbb{N}$.

**Definition 0.9.** $E \subset \mathbb{N}$ is said to be syndetic if there exists $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$E \cap \{n, n + 1, \ldots, n + k\} \neq \emptyset$$

If $T$ is a homeomorphism of a compact metric space $X$, a point $x \in X$ is called a uniformly recurrent point if for any open neighborhood $U$ of $x$, the set $\{n : T^n x \in U\}$ is syndetic.

**Proposition 0.10.** (Birkhoff) If $T$ is a homeomorphism of a compact metric space $X$, then there exists a uniformly recurrent point $x \in X$.

**Proposition 0.11.** Suppose $X$ is a compact metric space and $T : X \to X$ a homeomorphism such that $x \in X$ is uniformly recurrent. Then the orbit closure $\overline{\{T^n x : n \in \mathbb{Z}\}}$ is a minimal $T$-invariant closed subset of $X$.  

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For proofs see, for example, [Furstenberg, 1981], p.29. We now have the following characterization of sets of topological recurrence which is the topological analog of Proposition 0.5.

**Proposition 0.12.** Suppose \( E \subset \mathbb{N} \). Then \( E \) is a set of topological recurrence if and only if \( E \) is chromatically intersective.

**Proof.** Suppose \( E \) is a set of topological recurrence. Let \( X_k = \{1, \ldots, k\}^\mathbb{Z} \) with the product topology, a compact metrizable space. Define \( T : X_k \to X_k \) by \((T\phi)_n = \phi_{n+1}\). Suppose we are given a \( k \)-coloring \( N = C_1 \cup \cdots \cup C_k \). Let \( x \in X_k \), be the point with \( x_n = j \) precisely when \( n \in C_j \). Let \( X_x = \{T^n x : n \in \mathbb{Z}\} \). \( X_x \) is closed and \( T \)-invariant so there exists a uniformly recurrent point \( \psi \in X_x \). By the foregoing proposition \( T \) acts minimally on \( X_\psi \). If \( U = \{\gamma \in X_\psi : \gamma_0 = \psi_0\} \), then \( U \) is an open neighborhood of \( \psi \) and there exists \( r \in E \) such that \( U \cap T^r U \neq \phi \). Therefore \( \gamma_0 = \gamma_r \) for some \( \gamma \in X_x \). Since \( \gamma \in X_x \), this in turn implies that \( x_n = x_{n+r} \) for some \( n \), so \( n, n + r \in C_x \), and \( E \) is a set of chromatic recurrence.

The converse is from [Forrest, 1990]. Suppose \( E \) is chromatically intersective. If now \((X, T)\) is minimal with \( U \subset X \) open, we have (by minimality) \( \bigcup_{i=1}^k T^i U = X \). By compactness, there exists \( k \) such that \( \bigcup_{i=1}^k T^i U = X \). Let \( x \in X \) and consider a \( k \)-coloring of \( N \) such that \( n \in C_i \) implies \( T^n x \in T^i U \). For some \( r \in E \), and some \( i \), and some \( n \in C_i \), \( n + r \in C_i \), therefore \( T^n x \) and \( T^{n+r} x \) lie in the same shift of \( U \). It follows that \( U \cap T^r U \neq \phi \) and \( E \) is a set of topological recurrence.

The first of three results we will prove is Theorem 1.2 below, which was first proved by Kriz ([Kriz, 1987].) Because of Propositions 0.5 and 0.12 it implies that sets of topological recurrence need not be sets of measure theoretic recurrence. His treatment of the problem was in graph theoretic terms, and resolved a question asked by V. Bergelson in 1985 (see [Bergelson, 1987] for the details of Bergelson’s conjecture.) Forrest gives another proof ([Forrest, 1990]) where he also observes that the result also resolves negatively the following question of Furstenberg ([Furstenberg, 1981], p.76.)

**Question 0.12.** If \( S \) is a subset of the natural numbers \( \mathbb{N} \) with positive density, does there necessarily exist a syndetic set \( W \) such that \( W - W \subset S - S \)?

**Answer.** No. By Theorem 1.2 we have a set \( R \) which is a set of chromatic intersectivity and a set \( S \subset \mathbb{N} \) of positive density such that \( R \cap (S - S) = \phi \). For any syndetic \( W \), \( R \cap (W - W) \neq \phi \) since finitely many shifts of \( W \) color \( \mathbb{N} \) and each of these shifts has the same difference set as \( W \). It follows that \( W - W \) cannot be contained in \( S - S \).

1. All of the proofs presented here depend heavily on an elegant idea of Rusza’s (indeed the proof of Theorem 1.2 follows [Rusza, 1987] more or less.
Proposition 1.1. ([Lovász, 1978]) Let $E$ be the family of $r$-element subsets of $\{1,2,\cdots,2r+k\}$. Given any $k$-coloring of $E$, there exists a disjoint pair of elements from $E$ which are of the same color.

Theorem 1.2. ([Kriz, 1987]) For every $\epsilon > 0$ there exists $A \subset \mathbb{N}$ such that $\overline{d}(A) > \frac{1}{2} - \epsilon$, and $C \subset \mathbb{N}$, which is chromatically intersective, that satisfy $(A - A) \cap C = \emptyset$. (Therefore $C$ is not density intersective.)

Proof. We claim that for every $\epsilon > 0, k \in \mathbb{N}$ there exist natural numbers $M, N, A' \subset \{0,1,2,\ldots,M(N-1)\}$, $B' \subset \{0,1,2,\ldots,M(N-1)\}$, and $C' \subset \{0,1,2,\ldots,M-1\}$, such that $C'$ is $k$-intersective, $(A' + C') \cap A' = \emptyset$, $(B' + C') \cap B' = \emptyset$, $\nu(A') > \frac{1}{2} - \epsilon$, and $\nu(B') > \frac{1}{2} - \epsilon$. (Here $\nu$ is normalized counting measure on $\{0, \ldots, MN-1\}$.)

We will prove the claim. Let $r$ and $N \in \mathbb{N}$ be large and let $p_1, \ldots, p_{2r+k}$ be large odd primes (how large will be specified shortly.) Putting $M = p_1 \cdots p_{2r+k}$ and taking 0 to be neither odd nor even let

\[ A' = \{ a : M \leq a < (N-1)M \text{ and } a \pmod{p_i} \text{ is even for } < r \text{ indices and odd for all other } i. \} \]

\[ B' = \{ b : M \leq b < (N-1)M \text{ and } b \pmod{p_i} \text{ is odd for } < r \text{ indices and even for all other } i. \} \]

We require $N, r$, and the $p_i$'s to be so large that $\nu(A') > \frac{1}{2} - \epsilon$ and $\nu(B') > \frac{1}{2} - \epsilon$. Note that if $a \in (A' \cup B')$ then $a \not\equiv 0 \pmod{p_i}$, $1 \leq i \leq 2r+k$. We now let

\[ C' = \{ c : 0 \leq c < M \text{ and } c \pmod{p_i} \text{ is } 1 \text{ or } p_i - 1 \text{ for } \geq 2r \text{ indices } i. \} \]

If $a \in A'$ and $c \in C'$, then $(a+c) \pmod{p_i}$ is even or zero for at least $r$ indices $i$, $1 \leq i \leq 2r+k$, hence $(A' + C') \cap A' = \emptyset$. Similarly $(B' + C') \cap B' = \emptyset$.

We need to show that $C'$ is $k$-intersective. Let

\[ D = \{ d : 0 \leq d < M - 1, d = 2 \pmod{p_i} \text{ for exactly } \}
\]

\[ r \text{ indices } i \text{ and 1 for all other } i\} \]

$D$ is in obvious $1-1$ correspondence with the family $E$ of $r$-element subsets of $\{1,2,\cdots,2r+k\}$. Therefore, by Proposition 1.1, given any $k$-coloring of $D$, there exists a monochromatic pair $d_1 > d_2$ such that $d_1 \pmod{p_i} = 2 = d_2 \pmod{p_i}$ never occurs, $1 \leq i \leq 2r+k$. It follows that $(d_1 - d_2) \pmod{p_i}$ is 1 or $p_i - 1$ for 2r indices, that is, $d_1 - d_2 \in C'$ and $C'$ is $k$-intersective. This establishes our claim. We note that $A' + C', B' + C' \subset \{0,1,\cdots,MN-1\}$.

Let $\epsilon > 0$. Let $(\epsilon_k)_{k=1}^\infty$ converge to zero very quickly, (how quickly will be specified shortly.) For every $k$ and we have numbers $M_k$, $N_k$, and sets exactly.) All proofs of this theorem seem to make use of the following result of Lovász, which had been conjectured by Kneser.
If in fact for all such $A_k$, $B_k$, and $C_k$ having the properties stated in the previous paragraph (with $\epsilon_k$ instead of $\varepsilon$) Notice each $n \in \mathbb{N}$ is uniquely expressable as a sum

$$n = a_1 + M_1 N_1 a_2 + M_1 N_1 M_2 N_2 a_3 + \cdots + M_1 N_1 \cdots M_i N_i a_{i+1}$$

where $a_i \in \{0, 1, 2, \cdots, M_i N_i - 1\}$, $1 \leq i \leq l + 1$.

The $(\varepsilon_k)$ are chosen so that one of the sets

$$A_1 = \{a_1 + M_1 N_1 a_2 + M_1 N_1 M_2 N_2 a_3 + \cdots + M_1 N_1 \cdots M_i N_i a_{i+1} : a_i \in A_i$$

for an even number of $i$ and $a_i \in B_i$ or $a_i = 0$ for all other $i\}$$

$$A_2 = \{a_1 + M_1 N_1 a_2 + M_1 N_1 M_2 N_2 a_3 + \cdots + M_1 N_1 \cdots M_i N_i a_{i+1} : a_i \in A_i$$

for an odd number of $i$ and $a_i \in B_i$ or $a_i = 0$ for all other $i\}$$

has upper density $> \frac{1}{2} - \varepsilon$. Let $A$ be that set. Let

$$C = C_1 \cup M_1 N_1 C_2 \cup \cdots$$

$C$ is $k-$intersective for all $k$. (It is routine to show that if $J$ is $k$-intersective then $z\cdot J$ is for any integer $z$.) Also $(A + C) \cap A = \phi$. To see this, suppose that

$$n = a_1 + M_1 N_1 a_2 + \cdots + M_1 N_1 \cdots M_i N_i a_{i+1} \in A$$

and

$$c = M_1 N_1 \cdots M_j N_j c_{j+1} \in C$$

now if $a_{j+1} = 0$ then $a_{j+1} + c_{j+1}$ is non-zero and in neither $A_{j+1}$ nor $B_{j+1}$, so $n + c \notin A$. If on the other hand $a_{j+1} \in \alpha_{j+1}$ ($B_{j+1}$), then $a_{j+1} + c_{j+1} \notin A_{j+1}$ ($B_{j+1}$) and again $n + c \notin A$. This completes the proof of Theorem 1.2.

2. We prove in this section two results of Forrest, which again provided answers to questions of Bergelson.

**Definition 2.1.** A subset $C \subseteq \mathbb{N}$ is called a set of strong recurrence if, whenever $B \subseteq \mathbb{N}$ with $\overline{d}(B) > 0$,

$$\limsup_{c \in C, c \to \infty} \overline{d}((c + B) \cap B) > 0$$

If in fact for all such $B$

$$\limsup_{c \in C, c \to \infty} \overline{d}((c + B) \cap B) \geq (\overline{d}(B))^2$$

then $C$ is called set of nice recurrence.

By Theorem 2.5 below these two properties are not equivalent. Bergelson proved ([Bergelson, 1985]) that if $D$ is a set of strong recurrence and
A ⊂ N² has positive density then there exists an infinite B ⊂ D such that B × B ⊂ A - A. He then asked whether there were sets of (measure-theoretic) recurrence which are not sets of strong recurrence. If not, then the hypotheses of the aforementioned result could be weakened. Forrest showed however that there are such sets.

Our proof of Theorem 2.4 differs from Forrest’s in that it uses products of cyclic prime-order groups similar to that used in the proof of Theorem 1.2. We do follow Forrest in the use of the following lemmas.

**Lemma 2.2.** ([Kleitman, 1966]) If C ⊂ {0, 1}²M, and for each a, b ∈ C, aᵢ = bᵢ for at least 2J indices i, then

\[ |C| \leq \sum_{i \leq M-J} \binom{2M}{i} \]

**Lemma 2.3.** (Central Limit Theorem) If (X, μ) is a probability space, Aᵢ ⊂ X, i ∈ N are independent subsets of measure α, 0 < α < 1, then for every real number a

\[ \lim_{k \to \infty} \mu \left( \left\{ x : \sum_{i=1}^{k} 1_{A_i}(x) - k\alpha \leq a\sqrt{k\alpha(1-\alpha)} \right\} \right) = F(a) \]

where \( F(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-t^2} dt \) is the distribution function determined by the standard normal distribution. F is a continuous and increasing, with F(0) = 1/2, F(a) → 0 as a → -∞, and F(a) → 1 as a → ∞. Another form for F(a) is

\[ \lim_{k \to \infty} \mu^k \left( \left\{ (x_1, x_2, \cdots, x_k) \in X^k : \sum_{i=1}^{k} 1_{A_i}(x_i) - k\alpha \leq a\sqrt{k\alpha(1-\alpha)} \right\} \right) \]

**Theorem 2.4.** ([Forrest, 1991]) There exists a set of recurrence C ⊂ N which is not a set of strong recurrence.

**Proof.** Let T be the unit circle in the complex plane. Let μ be the usual Lebesgue probability measure on T. Then μ^k is the usual measure on the k-torus T^k. We claim that

1. For all M > 0 and ε > 0, there exist arbitrarily large k, N, and J, with J ≈ M√Nk, and W ⊂ T^k, such that \( \mu^k(W) = \frac{1}{2} \) and for all \( r \in R \), where

\[ R = \{(x_1, x_2, \cdots, x_k) \in T^k : |\arg x_i - \pi| \leq \frac{\pi}{2N} \text{ for at least } k - 2J \text{ indices } i\} \]
we have $\mu^k((W r) \cap W) < 3\epsilon$. $R$ has the property that for any measurable subset $B \subset T^k$ with $BB^{-1} \cap R = \phi$, we have

$$\mu^k(B) \leq \frac{1}{2^{2Nk}} \sum_{i \leq Nk-J} \binom{2Nk}{i} \quad (2.2)$$

By Lemma 2.3 (with $\alpha = \frac{1}{4}$) the right-hand side is asymptotically $F(\frac{j}{\sqrt{k}})$, which will go to zero as $\frac{j}{\sqrt{k}} \to \infty$, or as $M \to \infty$.

We now make some observations before proving (1). For all $N$ there exists a measure-preserving map $g_N: T \to \{0,1\}^{2N}$ such that if $(g_N(x))_i \neq (g_N(y))_i$ for every $i, 1 \leq i \leq 2N$, then $|\arg xy^{-1} - \pi| \leq \frac{\pi}{2N}$. It follows that if $|\arg xy^{-1} - \pi| > \frac{\pi}{2N}$ then $(g_N(x))_i = (g_N(y))_i$ for at least one $i$. (The required maps are pie partitions with opposite slices going to reversed bit patterns. For example, if $N = 1$ send the first quadrant to $(1,1)$, the second to $(1,0)$, the third to $(0,0)$, and the fourth to $(0,1)$.) Let

$$h_{N,k}: T^k \to \{0,1\}^{2Nk} = (\{0,1\}^{2N})^k$$

be the map induced by $g_N$. Now, for $x, y \in T^k$, if $(h_{N,k}(x))_i \neq (h_{N,k}(y))_i$ for at least $2Nk - 2J$ indices $i$, then $|\arg xy^{-1}_{j_i} - \pi| \leq \frac{\pi}{2N}$ for at least $k - 2J$ indices $j_i$, hence $xy^{-1} \in R$.

We verify (2.2). Let $B \subset T^k$. If $BB^{-1} \cap R = \phi$ then for all $x = (x_i), y = (y_i) \in B$, $|\arg x_i y_i^{-1} - \pi| > \frac{\pi}{2N}$ for at least $2J$ indices $i, 1 \leq i \leq k$, hence $(h_{N,k}(x))_i = (h_{N,k}(y))_i$ for at least $2J$ indices $i, 1 \leq i \leq 2Nk$, and by Kleitman’s lemma

$$|h_{N,k}(B)| \leq \sum_{i \leq Nk-J} \binom{2Nk}{i}$$

But $h_{N,k}$ is measure-preserving, so (2.2) is satisfied.

We now complete the proof of (1). By Lemma 2.3 we may pick $t > 0$ small enough that if

$$W_k = \{x \in T^k : \mathrm{Re} \ x_i \geq 0 \text{ for at least } \frac{k}{2} + t\sqrt{k} \text{ indices } i\} \quad (2.3)$$

then $\lim_{k \to \infty} \mu^k(W_k) > \frac{1}{2} - \epsilon$. Fix a large $N$ to be determined and let $k, J \to \infty$ according to $J \approx M \sqrt{Nk}$. In particular we assume that $J < \frac{k}{4}$. For each choice of $k, J$, consider the set $R$ of (2.1), $\sup_{r \in R} \mu^k(W_k \cap (W_k r))$ is attained for such an $r$ with $r_i = 1$ for $2J$ indices $i$, and $|\arg r_i - \pi| = \frac{\pi}{2N}$ for the other $k - 2J$ indices. (Actually $\mu^k(W_k \cap (W_k x))$ is increasing in $\mathrm{Re}$
For such an \( r \), and \( x \in W_k \cap (W_k r) \), we have \( x \in W_k \) and \( x r^{-1} \in W_k \). A look at (2.3) shows us that one of the following must occur:

(a) \( \text{Re } x_i > 0 \) for at least \( J + \frac{1}{2}t \sqrt{k} \) indices \( i \), \( 1 \leq i \leq 2J \), or

(b) \( \text{Re } x_i r_i^{-1} > 0 \) for at least \( J + \frac{1}{2}t \sqrt{k} \) indices \( i \), \( 1 \leq i \leq 2J \), or

(c) \( \text{Re } x_i > 0 \) for at least \( \frac{k}{2} - J + \frac{1}{2}t \sqrt{k} \) indices \( i \), \( 2j + 1 \leq i \leq k \), and the same is true of \( \text{Re } x_i r_i^{-1} \).

Since \( J \) increases as \( \sqrt{k} \) and \( t > 0 \), the probability of (a), which is to say the measure of the set of points \( x = (x_i)_{i=1}^k \in T^k \) satisfying (a), goes to zero by Lemma 2.3. The same is true for (b). As for (c), when \( 2J + 1 \leq i \leq k \), the probability that \( \text{Re } x_i > 0 \) and \( \text{Re } x_i r_i^{-1} > 0 \) is \( 2^{-(N+1)} \), as is the probability that they are both negative. (c) implies that the former happens for at least \( t \sqrt{k} \) more indices \( i \) than the latter. By Lemma 2.3 the probability of that happening for \( x \in T^k \) can be made arbitrarily small by choosing \( N \) large enough. In other words, \( N \) can be chosen so that for large enough \( k \) and \( J \approx M \sqrt{Nk} \), \( \mu^k (W_k \cap (W_k r)) \) is small for all \( r \in R \). The proof of claim (1) is completed by picking \( k \) odd and letting

\[
W = \{ x \in T^k : \text{Re } x_1 > 0 \text{ for at least } \frac{k}{2} \text{ indices } i \}
\]

Thus far it has been convenient to work in the torus \( T^k \) because of the naturalness of applying Lemma 2.3. We will denote by \( C_q \) the cyclic group \( \{0, \ldots, q-1\} \) under addition mod \( q \). Let \( \mu_q \) be normalized counting measure on \( C_q \). What we really will need is the following:

(2) For every \( \epsilon > 0 \), there exists \( M = p_1p_2 \cdots p_k \), where \( p_i \) are distinct primes, \( 1 \leq i \leq k \), and \( R \subset C_M \), such that if \( A \subset C_M \), \( \mu_M(A) > \epsilon \), then there exists \( r \in R \) such that \( \mu_M (A \cap (A + r)) > 0 \) and \( W \subset C_M \), \( \mu_M (W) \approx \frac{1}{q} \), such that for every \( r \in R \), \( \mu_M (W \cap (W + r)) < \epsilon \) and \( \mu_M (W^c \cap (W^c + r)) < \epsilon \).

(2) is true for the same reasons as (1). Instead of working in \( T^k \), however, one works in the natural subgroup \( C_{p_1} \times \cdots \times C_{p_k} \subset T^k \), where \( p_1, \ldots, p_k \) are large enough. This subgroup is of course isomorphic to \( C_M \).

We now complete the proof of the theorem. For a sequence \( \{ \epsilon_k \} \) converging to zero (at a rate to be determined shortly), let \( M_k, W_k \), and \( R_k \) be as in (2) and let \( N_k \) be so large that \( \mu_{M_kN_k} (A_k) > \frac{1}{2} - \epsilon_k \) and \( \mu_{M_kN_k} (B_k) > \frac{1}{2} - \epsilon_k \), where

\[
A_k = \{ a : M_k \leq a < (N_k - 1)M_k \text{ and } a \text{(mod } M_k) \in W_k \}
\]

\[
B_k = \{ b : M_k \leq b < (N_k - 1)M_k \text{ and } b \text{(mod } M_k) \in W_k^c \}
\]
Let \( C_k = R_k \) considered now as a subset of \( \mathbb{N} \). For each \( c \in C_k \), we have 
\[
\mu_{M_kN_k}(A_k \cap (A_k + c)) < \varepsilon_k \text{ and } \mu_{M_kN_k}(B_k \cap (B_k + c)) < \varepsilon_k.
\]
The \( \{\varepsilon_k\} \) are chosen so that the following set has upper density greater than \( 1 - \varepsilon \):
\[
\{\alpha_1 + M_1N_1\alpha_2 + \cdots + M_1N_1 \cdots M_kN_k\alpha_{k+1} : \alpha_i \in A_i \cup B_i \cup \{0\}\}
\]
Let \( F \) be the subset of the above set consisting of those numbers that have \( \alpha_i \in A_i \) for an even number of indices \( i \). Then \( \overline{d}(F) > \frac{1}{2} - \varepsilon \). Now let
\[
C = C_1 \cup M_1N_1C_2 \cup M_1N_1M_2N_2C_3 \cup \cdots
\]
We claim that \( C_i \) has the property that if \( D \subset \mathbb{N}, \overline{d}(D) > \varepsilon_i \), then there exists \( r \in C_i \) with \( \overline{d}((D + r) \cap D) > \frac{\varepsilon_i}{M_i} \). Suppose then that \( \overline{d}(D) > \varepsilon_i \). Then there exists an sequence \( (a_t) \subset \mathbb{N} \) satisfying \( a_{t+1} - a_t > M_i \), with \( \overline{d}((a_t)) > \frac{\varepsilon_i}{M_i} \), such that for each \( t \),
\[
\left| D \cap \{a_t, a_t + 1, \cdots, a_t + M_i - 1\} \right| > \varepsilon_iM_i
\]
Recall that \( C_i = R_i \) viewed as a subset of \( \mathbb{N} \). A glance at the definition of \( R_i \) shows us that \( r \in C_i \) if and only if \( M_i - r \in C_i \). This fact, coupled with the intersectivity property of \( R_i \), shows there exist \( a, b \in \{0, 1, \cdots, M_i - 1\} \), with \( a < b \), such that \( a + b \in D \), \( a + b \in D \), and \( r = b - a \in C_i \). It follows from the first observation that \( a + b \in D \cap (D + r) \). Therefore we have
\[
\overline{d}\left( \bigcup_{r \in C_i} (D \cap D + r) \right) > \overline{d}((a_t)) > \frac{\varepsilon_i}{M_i}
\]
Hence for some \( r \in C_i \) we have \( \overline{d}((D + r) \cap D) > \frac{\varepsilon_i}{M_i} \), establishing our claim. One routinely verifies that the claim is true for \( C_i \) replaced by any constant multiple of \( C_i \). From this fact it follows that \( C \) is a set of recurrence.

However, by construction
\[
\lim_{c \in C, c \to \infty} \overline{d}(F \cap (F + c)) = 0
\]
because, if \( c = M_1N_1 \cdots M_{n-1}N_{n-1}c_n \), with \( c_n \in C_n \), then the only elements of \( F \cap (F + c) \) are those numbers
\[
\alpha_1 + M_1N_1\alpha_2 + \cdots + M_1N_1 \cdots M_kN_k\alpha_{k+1}
\]
with \( \alpha_n \in A_n \) and \( \alpha_n - c_n \in A_n \) (or else both these in \( B_n \)). The set of natural numbers with this property is less than two times the density of
$A_n \cap (A_n + e_n)$ in $\{0, 1, \cdots, M_nV_n - 1\}$, that is, less than $2e_n$. Hence $C$ is not a set of strong recurrence. This completes the proof of Theorem 2.4.

Finally we have another result of Forrest.

**Theorem 2.5.** ([Forrest, 1990]) There exists a set $\tilde{C} \subseteq \mathbb{N}$ which is a set of strong recurrence but not a set of nice recurrence.

**Proof:** Let us return to the notation of the proof of Theorem 2.4 and let

$$\tilde{C}_1 = C_1, \tilde{C}_2 = C_1, \tilde{C}_3 = C_2, \tilde{C}_4 = C_1, \tilde{C}_5 = C_2, \tilde{C}_6 = C_3, \tilde{C}_7 = C_1, \text{ etc.}$$

Define similarly $\tilde{N}_k, \tilde{M}_k, \tilde{A}_k$, and $\tilde{B}_k$. Following the construction of the previous proof with these new sets we obtain $\tilde{C}$ and $\tilde{F}$. According to the claim at the end of the proof of Theorem 2.4, $\tilde{C}$ is a set of strong recurrence. $\tilde{C}$ is not, however, a set of nice recurrence, since $\tilde{d}(\tilde{F} \cap (\tilde{F} + c)) < 2e_1$ for all $c \in \tilde{C}$, just as at the end of the proof of Theorem 2.4.

**References**


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