D sets and IP rich sets in Z

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1. Introduction.

In this paper we will be dealing with the space of ultrafilters on $\mathbf{Z}$, endowed with its usual algebraic structure and topology. A standard background reference is [HS].

A *filter* on $\mathbf{Z}$ is a nonempty set p of subsets of $\mathbf{Z}$ that is closed under finite intersections and supersets and does not contain $\emptyset$. An *ultrafilter* is a maximal filter, that is a filter not properly contained in another filter. We denote the set of ultrafilters on $\mathbf{Z}$ by $\beta\mathbf{Z}$, and endow $\beta\mathbf{Z}$ with the topology generated by the (closed, as well as open) sets $\hat{A} = \{p \in \beta\mathbf{Z} : A \in p\}$. With this topology, $\beta\mathbf{Z}$ becomes a compact Hausdorff space.

Identifying $z \in \mathbf{Z}$ with the *principal* ultrafilter $e(z) = \{A \subset \mathbf{Z} : z \in A\}$, $\beta\mathbf{Z}$ becomes a representation of the Stone-Čech compactification of $\mathbf{Z}$. Now there is a unique associative extension to $\beta\mathbf{Z}$ of the operation $+$ on $\mathbf{Z}$ having the property that for every $q \in \beta\mathbf{Z}$, the function $p \to p + q$ is continuous. (Thus making $(\beta\mathbf{Z}, +)$ a *compact right topological semigroup.*) There are several ways to describe this extension; we will content ourselves with the classical one, i.e.

$$A \in p + q \Leftrightarrow \{x \in \mathbf{Z} : A - x \in q\} \in p.$$

According to a theorem of Ellis [E], any compact right topological semigroup has idempotents. It is easy to see that if $p \in \beta\mathbf{Z}$ is idempotent (that is, if $p = p + p$) then any member of p is an *IP set*, that is, a set that contains the set of finite sums of some sequence:

$$FS\big(\langle x_i\rangle_{i=1}^{\infty}\big) = \big\{x_{i_1} + x_{i_2} + \cdots + x_{i_k} : i_1 < i_2 < \cdots < i_k\big\}.$$

Conversely, any IP set is a member of some idempotent ultrafilter. We call a set $A \subset \mathbf{Z}$ *IP** if it belongs to *every* idempotent ultrafilter p. (Equivalently, if A^c fails to be IP.) Note that as $\{0\}$ is an IP set, every IP* set contains $\{0\}$. Some authors require that IP sets be infinite. We shall call infinite IP sets *non-trivial.*

Recall that the *upper Banach density* of a set $A \subset \mathbf{Z}$ is defined as

$$d^*(A) = \limsup_{N-M\to\infty} \frac{|A \cap \{M, M+1, \ldots, N-1\}|}{N-M}.$$

We will be concerned here with two density-related stengthenings of the IP set notion. The first, that of *D set*, was introduced in [BD].

Definition 1.1 An ultrafilter $p \in \beta\mathbf{Z}$ having the property that $d^*(A) > 0$ for every $A \in p$ is said to be *essential.* If p is an essential idempotent and $A \in p$, we say that A is a *D set.* If B^c is not a D set (equivalently, if B belongs to every essential idempotent), we say that B is a D^* set.

The second, that of *IP rich set*, was recently developed by V. Bergelson and A. Leibman, who have proved (unpublished) that certain return-times intersect all such sets.

Definition 1.2. A set $A \subset \mathbf{Z}$ is *IP rich*, or an *AIP set*, if $A \setminus E$ is an IP set for every $E \subset \mathbf{Z}$ with $d^*(E) = 0$. If $B^c \subset \mathbf{Z}$ is not IP rich (equivalently, if $B \cup E$ is IP* for some E with $d^*(E) = 0$) we say that B is *AIP**.

As *AIP** is supposed to stand for *almost IP**, we prefer *IP rich* to *AIP*.

IP* sets are AIP* and D*. Since there are zero density IP sets, not every IP set is a D set, from which it follows that not every D* set is IP*. It's also clear that not every AIP* set is IP*, and routine that every AIP* set is D*. (If B is AIP* then $B \cup E$ is a member of every idempotent for some zero density E. E doesn't belong to any essential idempotent, so B must belong to all of them.)

V. Bergelson (personal communication) asked whether every D* set is AIP*. In this paper we give a negative answer to this question. That is, we prove:

Main Theorem. There are D* subsets of $\mathbf{Z}$ that are not AIP*.

This yields the following proper containments:

$$IP^* \subsetneqq AIP^* \subsetneqq D^*, \text{ or } IP \supsetneqq AIP \supsetneqq D.$$

The proof, carried out in Section 3, proceeds via construction of an IP rich set that is not a D set. Workable characterizations of D sets and IP rich sets, which are of independent interest, are given in Section 2. Our equivalent condition for IP richness, which we call *FS tree richness*, already appears in the literature. In [HS, Theorem 20.17] it is shown to be a necessary property of D sets (making its non-sufficiency potentially interesting to a different crowd), while in [T] it is proved by elementary means to be a partition regular property. Our equivalent condition for D sets, meanwhile, is inspired by and comparable to a combinatorial characterization of *central sets* given in [HMS].

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2. Tree structure characterizations of IP rich sets and D sets.

In this section we give characterizations of D sets and IP rich sets. These are modeled on an elementary characterization of so-called *central sets* by Hindman, Maleki and Strauss in [HMS]. We begin with several definitions.

Definition 2.1. Let Ω be the set of finite sequences of integers, including the empty sequence.

Sometimes, we will want to include zero in our finite sums sets.

Definition 2.2 $FS_0\big(\langle x_i\rangle\big) = FS\big(\langle x_i\rangle\big) \cup \{0\}$.

The following definition will be instrumental in the inductive process whereby we construct IP rich sets.

Definition 2.3 If $A \subset \mathbf{Z}$ and $f = (x_1, \ldots, x_k) \in \Omega$, we say that A is *IP rich over f* if for every $E \subset \mathbf{Z} \setminus FS\big(\langle x_1, \ldots, x_k\rangle\big)$ with $d^*(E) = 0$ there exist non-zero $x_{k+1}, x_{k+2}, \ldots \in \mathbf{Z}$ such that $FS\big(\langle x_i\rangle_{i=1}^{\infty}\big) \subset A \setminus E$.

Here is a related notion. Recall that a non-trivial IP set is just an infinite IP set.

Definition 2.4. Let $F \subset \mathbf{Z}$ be a finite set. An *IP$_F$ set* is a set $R + F$, where R is a non-trivial IP set. A set $J \subset \mathbf{Z}$ is IP$^*_F$ if J intersects every IP$_F$ set non-trivially.

The following lemma is a generalization of the fact that for any IP* set B and any $n \in \mathbf{Z}$, the set $n\mathbf{Z} \cap B$ is again IP*.

Lemma 2.5. Let $F \subset \mathbf{Z}$ be a finite set. If $J \subset \mathbf{Z}$ is an IP$^*_F$ set and $n \in \mathbf{N}$ then $(n\mathbf{Z}+F)\cap J$ is IP$^*_F$ as well.

Proof. For every non-trivial IP set R,

$$R + F \not\subset J^c \Rightarrow R \not\subset \bigcap_{f\in F} (J^c - f).$$

Therefore

$$\{0\} \cup \bigcup_{f\in F} (J - f) = \{0\} \cup \Big(\bigcap_{f\in F} (J^c - f)\Big)^c$$

is IP*, which implies that

$$\{0\} \cup \Big(n\mathbf{Z} \cap \bigcup_{f\in F} (J - f)\Big)$$

is IP*. So, for every non-trivial IP set R there exist $r \in R$, $f \in F$, $z \in \mathbf{Z}$ and $j \in J$ such that $r = nz = j - f$, so that $r + f = nz + f = j$, whence $(R + F) \cap \big((n\mathbf{Z} + F) \cap J\big) \neq \emptyset$. □

We now move to our characterization of IP rich sets.

Definition 2.6. A set $A \subset \mathbf{Z}$ is *FS tree rich* if there is a subset $T \subset \Omega$ having the following properties:

I1. $() \in T$;
I2. If $f \in T$ then $d^*\big(B_f\big) > 0$, where $B_{(x_1,\ldots,x_k)} = \{x \in \mathbf{Z} : (x_1, \ldots, x_k, x) \in T\}$.
I3. If $(x_1, \ldots, x_k) \in T$ then $FS\big(\langle x_1, \ldots, x_k\rangle\big) \subset A$.

Hindman and Strauss show (cf. [HS, Theorem 20.17]) that FS tree richness is necessary for D sets. We establish now that FS tree richness is necessary (and sufficient) for IP richness.

Theorem 2.7. Let $A \subset \mathbf{Z}$. Then A is IP rich if and only if it is FS-tree rich.

Proof. We start with:

Claim. If A is IP rich over $(x_1, \ldots, x_k)$ then

$$B = \big\{x \in \mathbf{Z} \setminus FS\big(\langle x_1, \ldots, x_k\rangle\big) : A \text{ is IP rich over } (x_1, \ldots, x_k, x)\big\} \tag{2.1}$$

has positive upper Banach density.

Suppose **Claim** is false. Pick recalcitrant $(x_1, \ldots, x_k)$ and let $F = FS_0\big(\langle x_1, \ldots, x_k\rangle\big)$. We will construct a set $E \subset \mathbf{Z} \setminus FS\big(\langle x_1, \ldots, x_k\rangle\big)$ with $d^*(E) = 0$ such that $A^c \cup E$ is IP$^*_F$, which will yield a contradiction.

Let
$$K = \{x \in \mathbf{Z} \setminus FS(\langle x_1, \ldots, x_k \rangle) : FS(\langle x_1, \ldots, x_k, x \rangle) \subset A\}.$$
Let $\prec$ be a well-order on $\mathbf{Z}$. We will construct sequences $(k_x)_{x \in K \setminus B}$ and $(E'_x)_{x \in K \setminus B}$ (of numbers tending to ∞ and sets, respectively) satisfying the following:

(a) For every $x \in K \setminus B$ and every interval I with $|I| \geq k_x$,
$$\Big| I \cap \bigcup_{y \in K \setminus B, y \prec x} E'_y \Big| \leq \frac{|I|}{|x|+1}.$$

(b) For every $x \in K \setminus B$, $d^*(E'_x) = 0$.

(c) For every $x \in K \setminus B$, $E'_x \subset k_x \mathbf{Z}$.

(d) If $x, y \in K \setminus B$ with $y \prec x$ then $k_y | k_x$.

(e) For every $x \in K \setminus B$,
$$\bigcap_{y \in F_x} \big((A \setminus E'_x) - y \big) \setminus \{0\}$$
is not IP, where $F_x = FS_0(\langle x_1, \ldots, x_k, x \rangle)$.

(f) For every $x \in K \setminus B$, $E'_x \subset \mathbf{Z} \setminus FS(\langle x_1, \ldots, x_k \rangle)$.

Supposing that this construction has been carried out, let
$$E = B \cup \bigcup_{x \in K \setminus B} E'_x.$$

By (2.1) and (f), $E \subset \mathbf{Z} \setminus FS(\langle x_1, \ldots, x_k \rangle)$. Also $d^*(E) = 0$. To see this, note that if I_x are intervals with $|I_x| = k_x$ then by (c) and (d) at most one member of $\bigcup_{y \in K \setminus B, y \not\prec x} E'_y$ can belong to I_x, whereas by (a) and the fact that $d^*(B) = 0$ one has
$$\Big| I_x \cap \bigcup_{y \in K \setminus B, y \prec x} E'_y \Big| + |I_x \cap B| = |I_x| o(1).$$

Moreover $A^c \cup E$ is IP^*_F as desired. For if its complement $A \setminus E$ contains $R + F$ for some (non-trivial) IP set R then picking $x \in R$ and an IP set R' not having 0 as a member such that $R' + \{0, x\} \subset R$ one will have
$$R' \subset \bigcap_{y \in F_x} \big((A \setminus E) - y \big) \setminus \{0\}.$$

The latter set is therefore IP, but $x \in K \setminus B$ ($x \in K$ by definition and $x \notin E \supset B$) and $E'_x \subset E$, so by (e) it is *not* IP.

It remains to show that one can carry out the construction. Suppose $x \in K \setminus B$ and k_y, E'_y have been determined for all $y \in K \setminus B$ with $y \prec x$. Since $x \notin B$, A is not IP rich over $(x_1,\ldots,x_k,x)$, so there is a set $E_x \subset \mathbf{Z}\setminus FS\big(\langle x_1,\ldots,x_k,x\rangle\big)$ with $d^*(E_x)=0$ such that $A\setminus E_x$ contains no set of the form $FS\big(\langle x_i\rangle_{i=1}^{\infty}\big)$ with $x_{k+1}=x$ and x_i non-zero for $i\geq k+2$. In particular, $\bigcap_{y\in F_x}\big((A\setminus E_x)-y\big)\setminus\{0\}$ is not IP.

It is clear that we may choose k_x in conformity with (a) and (d). Now put

$$E'_x=\Big(k_x\mathbf{Z}\cap\bigcup_{y\in F_x}(E_x-y)\Big)\setminus\{0\}.$$

Note that (b) and (c) are satisfied, and since $0\notin E'_x$, (f) is as well provided k_x is large enough, which we may require. We now establish (e).

We know that $\bigcap_{y\in F_x}\big((A\setminus E_x)-y\big)\setminus\{0\}$ is not IP, so its complement

$$\{0\}\cup\bigcup_{y\in F_x}\big((A^c\cup E_x)-y\big)$$

is IP*. Thus

$$k_x\mathbf{Z}\cap\Big(\{0\}\cup\bigcup_{y\in F_x}\big((A^c-y)\cup(E_x-y)\big)\Big)$$

is IP*, so that the potentially larger

$$\{0\}\cup\bigcup_{y\in F_x}(A^c-y)\cup\big(k_x\mathbf{Z}\cap\bigcup_{y\in F_x}(E_x-y)\big)=\{0\}\cup\bigcup_{y\in F_x}(A^c-y)\cup E'_x$$

is IP* as well. This set is however contained in

$$\{0\}\cup\bigcup_{y\in F_x}\big((A^c\cup E'_x)-y\big),$$

which is therefore IP*, implying that its complement

$$\bigcap_{y\in F_x}\big((A\setminus E'_x)-y\big)\setminus\{0\}$$

is not IP, yielding (e) and establishing **Claim**.

In light of the above claim, it is now easy to check that

$$T=\{f\in\Omega : A \text{ is IP rich over } f\}$$

satisfies I1-I3 above.

Conversely, suppose that T satisfies I1-I3 and let $E\subset\mathbf{Z}$ with $d^*(E)=0$. We must show that $A\setminus E$ contains an IP set. Since $()\in T$, $d^*\big(\{x\in\mathbf{Z}:(x)\in T\}\big)>0$, and for all x in this set, $x\in A$. So we may choose x_1 such that $(x_1)\in T$ and $x_1\notin E$. Next we have $d^*\big(\{x\in\mathbf{Z}:(x_1,x)\in T\}\big)>0$, and for every x in this set, $\{x,x+x_1\}\subset A$. Since

$d^*\big(E\cup(E-x_1)\big)=0$, we may choose x_2 such that $(x_1,x_2)\in T$ and $x_2\notin E\cup(E-x_1)$. Note now that $FS\big(\{x_1,x_2\}\big)\subset A\setminus E$. It is clear that this process can be continued and will yield a sequence $\langle x_i\rangle_{i=1}^\infty$ for which $FS\big(\langle x_i\rangle_{i=1}^\infty\big)\subset A\setminus E$. $\square$

We next move to our elementary characterization of D sets. One will immediately see that it is similar to the FS-tree richness condition, but stronger, in that the intersection of the successor sets of any finite family of nodes must have positive upper Banach density.

Theorem 2.8. Let $A\subset\mathbf{Z}$. Then A is a D set if and only if there is a subset $T\subset\Omega$ having the following properties:

D1. $()\in T$;

D2. If $f_1,\dots,f_t\in T$ then $d^*\big(B_{f_1}\cap\cdots\cap B_{f_t}\big)>0$, where

$$B_{(x_1,\dots,x_k)}=\{x\in\mathbf{Z}:(x_1,\dots,x_k,x)\in T\}.$$

D3. If $(x_1,\dots,x_k)\in T$ then $FS\big(\langle x_1,\dots,x_k\rangle\big)\subset A$.

Proof. We will be using the standard fact that if p is idempotent and $A\in p$ then $A\in p+p$, i.e. $\{m:A-m\in p\}\in p$. Let p be an essential idempotent with $A\in p$. Let $A_{()}=A\cap\{m:A-m\in p\}$. For $x\in A_{()}$, let

$$A_{(x)}=A\cap(A-x)\cap\{m:(A\cap(A-x))-m\in p\}\in p.$$

Note that for such x, $x\in A$. Now for $y\in A_{(x)}$, let

$$A_{(x,y)}=A\cap(A-x)\cap(A-y)\cap(A-x-y)\cap\big\{m:\big(A\cap(A-x)\cap(A-y)\cap(A-x-y)\big)-m\in p\big\}\in p.$$

Note that for such x,y, $FS\big(\langle x,y\rangle\big)\subset A$. Now for $z\in A_{(x,y)}$ one defines $A_{(x,y,z)}\in p$, etc. Continuing in this fashion, one defines p-sets $\{A_f:f\in T\}$ for some set $T\subset\Omega$. Letting $B_{(x_1,\dots,x_k)}=\{x\in\mathbf{Z}:(x_1,\dots,x_k,x)\in T\}$ one has $B_{(x_1,\dots,x_k)}=A_{(x_1,\dots,x_k)}$, and D1-D3 above are satisfied.

Conversely, suppose that T satisfies D1-D3. By expanding T if necessary, we can assume that T satisfies:

D4. If $(x_1,\dots,x_k)\in T$ and $L_1,L_2,\dots,L_r$ are consecutive blocks of natural numbers whose union is $\{1,2,\dots,k\}$ then, letting $y_i=\sum_{j\in L_i}x_j$, one has $(y_1,\dots,y_r)\in T$.

To see this, note that once we include every such $(y_1,\dots,y_r)$ for $(x_1,\dots,x_k)$ originally in T, D4 will be already satisfied, and that after doing so $B_{(x_1,\dots,x_k)}\subset B_{(y_1,\dots,y_r)}$; that is, every set of successors is a superset of an original set of successors, so D2 (and obviously D3) will still be satisfied. Now let

$$S=\bigcap_{\substack{f\in T,E\subset\mathbf{Z},\\ d^*(E)=0}}\overline{(B_f\setminus E)}.$$

As the sets $B_f \setminus E$ have the finite intersection property, S is non-empty and of course closed. Moreover if $p \in S$ and $C \in p$ then $d^*(C) > 0$, as otherwise $(B_{()} \setminus C) \in p$, a contradiction. Also $A \in p$ for all $p \in S$. We claim that S is a semigroup and thus contains idempotents; such idempotents will be essential and will contain A, and this will complete the proof.

Let $p, q \in S$. We need to show that $p + q \in S$. Let $C \in p + q$ be arbitrary. It suffices to find $r \in S$ with $C \in r$. (If $p + q$ were not a member of the closed set S, one could find a basic neighborhood $\overline{C} = \{r : C \in r\}$ of $p + q$ disjoint from S.) In order to show this it is sufficient to show that $d^*(\bigcap_{i=1}^h B_{f_i} \cap C) > 0$ for every $f_1, \ldots, f_h \in T$, as then we can choose

$$r \in \bigcap_{\substack{f \in T, E \subset \mathbf{Z}, \\ d^*(E) = 0}} \overline{((B_f \cap C) \setminus E)}.$$

One has $\{x \in \mathbf{Z} : C - x \in q\} \in p$, so since $p \in S$, for every $f_1, \ldots, f_h \in T$,

$$d^*\Big(\big\{x \in \bigcap_{i=1}^h B_{f_i} : C - x \in q\big\}\Big) > 0.$$

Fix $f_i = (x_1^{(i)}, \ldots, x_{k_i}^{(i)}) \in T$, $1 \le i \le h$. We may choose $x \in \bigcap_{i=1}^h B_{f_i}$ with $C - x \in q$. Since $q \in S$,

$$\begin{aligned} &d^*\big(\{n \in \bigcap_{i=1}^h B_{(x_1^{(i)}, \ldots, x_{k_i}^{(i)}, x)} : C - x \in p_n\}\big) \\ =&d^*\big(\{n \in \bigcap_{i=1}^h B_{(x_1^{(i)}, \ldots, x_{k_i}^{(i)}, x)} : x + n \in C\}\big) \\ =&d^*\big(\bigcap_{i=1}^h B_{(x_1^{(i)}, \ldots, x_{k_i}^{(i)}, x)} \cap (C - x)\big) > 0, \end{aligned}$$

where p_n is the principal ultrafilter on n. Put another way,

$$d^*\big(\{n \in \bigcap_{i=1}^h B_{(x_1^{(i)}, \ldots, x_{k_i}^{(i)}, x)} : n + x \in C\}\big) > 0.$$

Observe now that (by D4)

$$x + \{n \in \bigcap_{i=1}^h B_{(x_1^{(i)}, \ldots, x_{k_i}^{(i)}, x)} : n + x \in C\} \subset \bigcap_{i=1}^h B_{f_i} \cap C,$$

which completes the proof. $\square$

As mentioned in the introduction, Towsner [T] has shown by an elementary argument that for any finite partition of $\mathbf{Z}$, some cell is FS tree rich. In light of the example given in the next section, it's natural to issue the following:

Challenge. Give an elementary argument that for any finite partition of $\mathbf{Z}$, some cell A supports a tree T satisfying D1-D3 above, i.e. is a D set.

3. An IP rich set that is not a D set and proof of Main Theorem.

Characterizations in place, we are now ready to construct the set that will enable us to prove our main theorem.

Theorem 3.1 There exists a set $A \subset \mathbf{Z}$ such that A is IP rich and A is not a D set.

Proof. Recall that a subset of $\mathbf{Z}$ is *thick* if it contains arbitrarily long intervals. It is an exercise that there exists a countable pairwise disjoint family $\{S_i : i \in \mathbf{N}\}$ of thick subsets of $\mathbf{N}$. We will be constructing countably many sets A_f of positive upper Banach density in this proof. Each of these will be assumed to be contained in a separate member of such a family. By an n-spaced subset of some S_i we mean a set $B \subset S_i \cap [n, \infty)$ having the property that if $x \in B$ and $0 < |x - y| < n$ then $y \in S_i \setminus B$.

Let $A_{()} \subset S_1$ be a set of odd numbers with $d^*(A_{()}) > 0$. Let x_1 be the least member of $A_{()}$. Choose m_1 with $2^{m_1} > x_1$ and let $A_{(x_1)}$ be a 2^{m_1+2}-spaced subset of S_2 consisting of numbers equal to $2^{m_1} \pmod{2^{m_1+1}}$ with $d^*(A_{(x_1)}) > 0$. Now pick the least member x_2 of $A_{()} \cup A_{(x_1)}$ not already used (i.e. not x_1). Suppose that x_2 comes from $A_{(x_1)}$. Choose $m_2 > m_1$ with $2^{m_2} > (x_1 + x_2)$ and let $A_{(x_1,x_2)}$ be a 2^{m_2+2}-spaced subset of S_3 consisting of numbers equal to $2^{m_2} \pmod{2^{m_2+1}}$ with $d^*(A_{(x_1,x_2)}) > 0$. Let x_3 be the least member of $A_{()} \cup A_{(x_1)} \cup A_{(x_1,x_2)}$ not already used. Say it comes from $A_{()}$. Choose $m_3 > m_2$ with $2^{m_3} > (x_1 + x_2 + x_3)$ and let $A_{(x_3)}$ be a 2^{m_3+2}-spaced subset of S_4 consisting of numbers equal to $2^{m_3} \pmod{2^{m_3+1}}$ with $d^*(A_{(x_3)}) > 0$.

Continue in this fashion; at the stage where we are ready to create a set within S_{k+1}, we let x_k be the least member of any of the sets constructed in previous stages that was not already used. Say it comes from a set $A_{(a_1,\ldots,a_t)}$. Choose $m_k > m_{k-1}$ with $2^{m_k} > (x_1 + \cdots + x_k)$ and let $A_{(a_1,\ldots,a_t,x_k)}$ be a 2^{m_k+2}-spaced subset of S_{k+1} consisting of numbers equal to $2^{m_k} \pmod{2^{m_k+1}}$ with $d^*(A_{(a_1,\ldots,a_t,x_k)}) > 0$.

We note that:

A. $A_{(a_1,\ldots,a_t,x_k)} + FS_0\big(\langle a_1, \ldots, a_t, x_k\rangle\big) + \{0, 1, \ldots, 2^{m_k}\} \subset S_{k+1}$.

B. Every member of $A_{(a_1,\ldots,a_t,x_k)}$ is divisible by 2^{m_k}.

C. No member of $A_{(a_1,\ldots,a_t,x_k)} + FS_0\big(\langle a_1, \ldots, a_t, x_k\rangle\big)$ is divisible by 2^{m_k+1}.

Let T be the set of $(a_1, \ldots, a_k) \in \Omega$ used as subscripts for sets $A_{(\cdot)}$ in this construction.

D. Letting $B_{(a_1,\ldots,a_k)} = \{x \in \mathbf{Z} : (a_1, \ldots, a_k, x) \in T\}$, one has $B_{(a_1,\ldots,a_k)} = A_{(a_1,\ldots,a_k)}$.

Next put

$$A = \bigcup_{(a_1,\ldots,a_k)\in T} \Big(A_{(a_1,\ldots,a_k)} + FS_0\big(\langle a_1, \ldots, a_k\rangle\big)\Big) = \bigcup_{(a_1,\ldots,a_k)\in T} FS\big(\langle a_1, \ldots, a_k\rangle\big).$$

Then I1-I3 above are plainly satisfied, so A is IP rich. We now turn to showing that A is not a D set.

E. If $(a_1, \ldots, a_t) \in T$ and $m \in \mathbf{N}$ then

(1) $4a_i \le a_{i+1}$, $1 \le i < t$.

(2) If $a_t \equiv 2^m \pmod{2^{m+1}}$ then $a_i \not\equiv 0 \pmod{2^m}$, $1 \le i < t$.

(3) If for some $1 \le i_1 < i_2 < \cdots < i_k \le t$ one has $\big(a_{i_1} + a_{i_2} + \cdots + a_{i_k}\big) \equiv 0 \pmod{2^m}$ then $a_{i_1} \equiv 0 \pmod{2^m}$. (Hence $a_{i_j} \equiv 0 \pmod{2^m}$, $1 \le j \le k$.)

(1) follows from the fact that $A_{(a_1,\ldots,a_j,x_k)}$ is $4(x_1 + \cdots + x_k)$-spaced. For (2), note that for some $i_1 < i_2 < \cdots < i_t$, $(a_1, \ldots, a_t) = (x_{i_1}, \ldots, x_{i_t})$. Since $x_{i_t} \in A_{(x_{i_1},\ldots,x_{i_{t-1}})}$, $x_{i_t} \equiv 2^{m_{i_{t-1}}} \pmod{2^{m_{i_{t-1}}+1}}$. This implies that $m = m_{i_{t-1}}$. Now use the fact that the sequence m_j increases with j. For (3), assume the negation and choose a shortest (i.e. minimum k, but note $k \ge 2$) counterexample. Then obviously $a_{i_k} \not\equiv 0 \pmod{2^m}$. Choose r such that $a_{i_k} \equiv 2^r \pmod{2^{r+1}}$. Then

$$\big(a_{i_1} + a_{i_2} + \cdots + a_{i_{k-1}}\big) \equiv 0 \pmod{2^r}$$

but $a_{i_1} \not\equiv 0 \pmod{2^r}$ (again, since m_j increases with j). So this is a shorter counterexample, which is a contradiction.

F. If $\langle a_i \rangle_{i=1}^{\infty}$ is a sequence having the property that $(a_1, \ldots, a_t) \in T$ for every $t \in \mathbf{N}$ then $d^*\big(FS(\langle a_i \rangle_{i=1}^{\infty})\big) = 0$.

This follows from **E** (1). For let $t \in \mathbf{N}$ and let I be any interval of length 4^t. Since $a_{t+1} \ge 4^t$, I contains at most one member of $x + FS(\langle a_i \rangle_{i=t+1}^{\infty})$ for any $x \in FS(\langle a_i \rangle_{i=1}^{t})$. Therefore, I contains no more than 2^t members of $FS(\langle a_i \rangle_{i=1}^{\infty})$.

G. For all $x, y \in A$, if there exists an IP set $R \subset \mathbf{N}$ with $R \cup (R + x) \cup (R + y) \subset A$ then there exists some $(a_1, \ldots, a_k) \in T$ such that $\{x, y\} \subset FS\big(\langle a_1, \ldots, a_k \rangle\big)$.

To see this, pick m such that 2^m is greater than $\max\{x, y\}$. Under the hypothesis about R, A must contain a configuration of the form $\{h2^m, h2^m + x, h2^m + y\}$. By definition of A, $h2^m$ is a member of some set $A_{(a_1,\ldots,a_t,x_k)} + FS_0\big(\langle a_1, \ldots, a_t, x_k \rangle\big)$. By **C** no member of that set is divisible by 2^{m_k+1}. This implies that $m \le m_k$, so that $\max\{x, y\} < 2^{m_k}$. Then by **A**, $\{h2^m, h2^m + x, h2^m + y\} \subset S_{k+1}$, which implies that in fact

$$\{h2^m, h2^m + x, h2^m + y\} \subset A \cap S_{k+1} = A_{(a_1,\ldots,a_t,x_k)} + FS_0\big(\langle a_1, \ldots, a_t, x_k \rangle\big).$$

But $A_{(a_1,\ldots,a_t,x_k)}$ is 2^{m_k+2}-spaced, $2^{m_k} > x_1 + \cdots + x_k$, $\max\{x, y\} < 2^{m_k}$, so for some $x_j \in A_{(a_1,\ldots,a_t,x_k)}$ one actually has

$$\{h2^m, h2^m + x, h2^m + y\} \subset \{x_j\} + FS_0\big(\langle a_1, \ldots, a_t, x_k \rangle\big) \subset FS\big(\langle a_1, \ldots, a_t, x_k, x_j \rangle\big).$$

Now write $(x_{i_1}, \ldots, x_{i_z}) = (a_1, \ldots, a_t, x_k, x_j)$ and suppose that $x_{i_1}, \ldots, x_{i_q}$ are not divisible by 2^m while $x_{i_{q+1}}, \ldots, x_{i_z}$ are; this is possible by **E** (2). By **E** (3), no member of $FS(\langle x_{i_1}, \ldots, x_{i_q} \rangle)$ is divisible by 2^m, so $h2^m \in FS\big(\langle x_{i_{q+1}}, \ldots, x_{i_z} \rangle\big)$. Now, every member of $FS\big(\langle x_{i_{q+1}}, \ldots, x_{i_z} \rangle\big)$ is divisible by $2^{m_{i_q}}$. On the other hand, by **C** $x_{i_{q+1}}$ is not divisible by $2^{m_{i_q}+1}$. Therefore $m_{i_q} \ge m$, whence $\max\{x, y\} < 2^{m_{i_q}}$. It's also the case (by stipulation; see the construction) that $x_{i_1} + \cdots + x_{i_q} < 2^{m_{i_q}}$. Now since $M + x' = h2^m + x$ for some $M \in FS\big(\langle x_{i_{q+1}}, \ldots, x_{i_z}\rangle\big)$ and $x' \in FS\big(\langle x_{i_1}, \ldots, x_{i_q}\rangle\big)$, $2^{m_{i_q}} | (h2^m - M) = (x' - x)$. So $x = x' \in FS\big(\langle x_{i_1}, \ldots, x_{i_q}\rangle\big)$. As a similar argument applies to y, we have $\{x, y\} \subset FS\big(\langle x_{i_1}, \ldots, x_{i_q}\rangle\big)$.

H. Suppose that $(x_{i_1}, \ldots, x_{i_k}), (x_{j_1}, \ldots, x_{j_t}) \in T$. If

$$\Big(FS\big(\langle x_{i_1}, \ldots, x_{i_k}\rangle\big) \setminus FS\big(\langle x_{i_1}, \ldots, x_{i_{k-1}}\rangle\big)\Big) \cap \Big(FS\big(\langle x_{j_1}, \ldots, x_{j_t}\rangle\big) \setminus FS\big(\langle x_{j_1}, \ldots, x_{j_{t-1}}\rangle\big)\Big)$$

is non-empty then $k = t$ and $i_s = j_s$, $1 \leq s \leq t$.

Note that, by construction, $(a_1, \ldots, a_j) \in T$ is uniquely determined by a_j. (If $a_j \in S_{k+1}$ then $a_{j-1} = x_k$. Now use induction.) So by symmetry we may assume that if there is a counterexample to **H** then there is a counterexample with $x_{i_k} < x_{j_t}$. But since $x_{i_k} \in A_{(x_{i_1}, \ldots, x_{i_{k-1}})}$, it has distance at least $4(x_1 + \cdots + x_{i_{k-1}})$ from any other x_i. It follows that every member of $FS\big(\langle x_{i_1}, \ldots, x_{i_k}\rangle\big)$ is less than x_{j_t}, a contradiction.

Suppose now that A is a D set. Then there is a tree $T' \subset \Omega$ which, together with its successor sets B'_f, satisfies D1-D3 above. In particular, for each $y, z \in B'_{()}$ there is some IP set $R \subset \mathbf{N}$ with $R \cup (R + y) \cup (R + z) \subset A$. By **G**, then, for every $y, z \in B'_{()}$ there there exists some $(a_1, \ldots, a_k) \in T$ such that $\{y, z\} \subset FS\big(\langle a_1, \ldots, a_k\rangle\big)$.

Consider now the map from $B'_{()}$ to T that sends $x \in B'_{()}$ to the unique (by **H**) $\pi(x) = (a_1, \ldots a_k) \in T$ having the property that $x = a_k + y$ for some $y \in FS_0\big(\langle a_1, \ldots, a_{k-1}\rangle\big)$. What **G** tells us is that for every $y, z \in B'_{()}$, either $\pi(y)$ is an initial segment of $\pi(z)$ or vice-versa. Since for any fixed y there can be only finitely many $z \in B'_{()}$ such that $\pi(z)$ is an initial segment of $\pi(y)$, the length of $\pi(y)$ as y ranges over the infinite set $B'_{()}$ is unbounded and there exists at least one infinite sequence $(a_1, a_2, \ldots)$ in the closure of $\pi(B'_{()})$ (topology of pointwise convergence). So $\pi(y)$ is an initial segment of $(a_1, a_2, \ldots)$ for every $y \in B'_{()}$ (otherwise we could find $z \in B'_{()}$ such that neither of $\pi(y), \pi(z)$ was an initial segment of the other). Therefore $B'_{()} \subset FS\big(\langle a_i\rangle_{i=1}^{\infty}\big)$, and since by **E** (1) $4a_i \leq a_{i+1}$ for every i, one has $d^*\big(FS\big(\langle a_i\rangle_{i=1}^{\infty}\big)\big) = 0$ by **F**, contradicting $d^*(B'_{()}) > 0$. □

Recall now from the introduction our main theorem, which we are ready to prove.

Main Theorem. There are D* subsets of **Z** that are not AIP*.

Proof of Main Theorem. Let A be the set constructed in the previous theorem. Then $\mathbf{Z} \setminus A$ is D* but not AIP*. □

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