Reading "A variant of the Hales-Jewett theorem" on its anniversary

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Dedicated to Neil Hindman on the occasion of his .65th centenary.

In a recent paper "A variant of the Hales-Jewett theorem", M. Beiglböck provides a version of the classic coloring result in which an instance of the variable in a word giving rise to a monochromatic combinatorial line can be moved around in a finite structure of specified type (for example, an arithmetic progression). We give an elementary proof and infinitary extensions.

0. Quantifying N.

In [B], M. Beiglböck proves a curious extension of the Hales-Jewett theorem, in which one of the occurrences of the variable in a variable word giving rise to a monochromatic combinatorial line is allowed to move around in a finite set of a predetermined class $\mathcal{P}$ (say, arithmetic progressions of fixed length). However, his proof is daunting in its use of heavy algebraic/topological machinery. Fortunately, standard combinatorial methods (the "lines imply spaces" paradigm) can be used to simplify the proof while upgrading the formulation (to one in which *all* occurrences of the variable may move).

Our method gives quantitative upper bounds in the case of, say, $\mathcal{P}$ the class of arithmetic progressions of a fixed length. (The original proof gives no bounds.) The first author has discussed this in [Bo]. Briefly, the proof utilizes one primitive recursion beyond that used in Shelah's proof of the Hales-Jewett theorem, yielding bounds in the 6th class $\mathcal{E}^6$ of the Grzegorczyk hierarchy; cf. [S].

We remark that one may derive (see [B] for details) from Beiglböck's theorem a partition version of a result of V. Bergelson (cf. [Be, Theorem 1.5]; also [BBHS, Corollary 4.3]) which states that for any finite partition of $\mathbf{N}$ and any $k \in \mathbf{N}$, there is a configuration of the form $\{b(a+id)^j : 0 \leq i, j < k\}$ contained in a single cell of the partition. We know of no previous elementary proof of this result.

Demonstratives: Neil Hindman's influence is everywhere evident in this paper, and it is in deference to him that it is (mostly) written in the ultrafilter style, using his preferred left/right conventions. Also, an anonymous referee suggested numerous corrections and improvements, including a strengthening of Theorem 3.3.

1. Words.

1.1. Let $N \in \mathbf{N}$ and consider a finite alphabet which we denote by $\{0, 1, \ldots, k\}$. A *word* of length N is a member of $\{0, 1, \ldots, k\}^N$; a *variable word* of length N is a member of $\{0, 1, \ldots, k, x\}^N \setminus \{0, 1, \ldots, k\}^N$, where x is not a member of the original alphabet. We

use the notation $w(x)$ for a variable word; in this case, for $t \in \{0, 1, \ldots, k\}$, $w(t)$ denotes the word that obtains when all occurrences of "x" in $w(x)$ are replaced by "t".

The Hales-Jewett coloring theorem [HJ] states that for every $k, r \in \mathbf{N}$, there exists $N \in \mathbf{N}$ such that if $\{0, 1, \ldots, k\}^N$ is r-colored, there is a variable word $w(x)$ of length N such that $\{w(t) : 0 \leq t \leq k\}$ is monochromatic.

1.2. Let $\mathcal{F}$ denote the family of finite, non-empty subsets of $\mathbf{N}$. For $\alpha, \beta \in \mathcal{F}$, we write $\alpha < \beta$ if $i < j$ for every $i \in \alpha$ and $j \in \beta$.

1.3. Let $w = u_1 u_2 \cdots u_N$ be a word or variable word, let $\alpha \in \mathcal{F}$ and suppose that $u_i = 0$ for every $i \in \alpha$. We denote by $w^\alpha(x)$ the variable word that results upon replacement of each u_i, $i \in \alpha$, by "x".

A subfamily $\mathcal{P}$ of $\mathcal{F}$ is *partition regular* if $\mathcal{P}$ contains no singletons and if for every finite coloring of $\mathbf{N}$ there is a monochromatic member of $\mathcal{P}$. Note that, by a routine compactness argument, partition regularity of $\mathcal{P}$ implies that once the number of colors is fixed, for any $N_0 \in \mathbf{N}$ there is some $N \in \mathbf{N}$ such that for any coloring of $\{N_0 + 1, \ldots, N_0 + N\}$ in that fixed number of colors, there is a monochromatic member of $\mathcal{P}$.

1.4. Beiglböck's "variant" of Hales-Jewett states (in the current terminology) that for every $k \in \mathbf{N}$ and every partition-regular subfamily $\mathcal{P}$ of $\mathcal{F}$, if $\bigcup_{N=1}^{\infty} \{0, 1, \ldots, k\}^N$ is finitely colored, there are $N \in \mathbf{N}$, a (possibly variable) word $w = u_1 \ldots u_N$ and some $\alpha \in \mathcal{P}$ with $u_i = 0$ for all $i \in \alpha$, such that

$$\left\{ w^{\{j\}}(t) : j \in \alpha, 0 \leq t \leq k \right\}$$

is monochromatic.

1.5. The proof of Theorem 1.4 in [B] is non-elementary. We give now a slightly more powerful formulation, together with an elementary proof.

Theorem. Let $k, r \in \mathbf{N}$ and suppose $\mathcal{P}$ is a partition-regular family of finite subsets of $\mathbf{N}$. There exists $N = N(k, r, \mathcal{P}) \in \mathbf{N}$ such that if $\{0, 1, \ldots, k\}^N$ is r-colored, then there exist $l \in \mathbf{N}$, $\beta_i \in \mathcal{P}$, $1 \leq i \leq l$, with $\beta_1 < \beta_2 < \cdots < \beta_l < \{N + 1\}$, and $w = u_1 u_2 \cdots u_N \in \{0, 1, \ldots k\}^N$ having the property that $u_i = 0$ for all $i \in \bigcup_{j=1}^{l} \beta_j$, such that

$$\left\{ w^{\{j_1, \ldots, j_l\}}(t) : j_i \in \beta_i, 1 \leq i \leq l, 0 \leq t \leq k \right\}$$

is monochromatic.

1.6. For the proof of Theorem 1.5, we require the following.

Lemma. Let $r, k, M \in \mathbf{N}$, and suppose $\mathcal{P}$ is a partition-regular family of finite subsets of $\mathbf{N}$. There exists $N \in \mathbf{N}$ such that for any r-coloring $c : \{0, 1, \ldots, k\}^N \to \{1, 2, \ldots, r\}$, there exist $\alpha_a \in \mathcal{P}$, $1 \leq a \leq M$, with $\alpha_1 < \alpha_2 < \cdots < \alpha_M$, having the property that for any $i_a, j_a \in \alpha_a$ and $t_a \in \{0, 1, \ldots, k\}$, $1 \leq a \leq M$, one has $c(u_1 u_2 \cdots u_N) = c(v_1 v_2 \cdots v_N)$, where $u_{i_a} = v_{j_a} = t_a$ and any u_l or v_l not so defined is "0". (That is, the color of a word having (potentially) non-zero entries $t_1, \ldots, t_M$, occuring at places $i_1, \ldots, i_M$ belonging to $\alpha_1, \ldots, \alpha_M$ respectively, depends on the t_a, but not on the i_a.)

Proof. Let N_1 be so large that for any $r^{(k+1)^M}$-coloring of $\{1, 2, \ldots, N_1\}$, there is a monochromatic member of $\mathcal{P}$. Let N_2 be so large that for any $r^{(k+1)^{N_1+M-1}}$-coloring of $\{N_1+1, N_1+2, \ldots, N_1+N_2\}$, there is a monochromatic member of $\mathcal{P}$. Having chosen $N_1, N_2, \ldots, N_{j-1}$, choose N_j so large that for any $r^{(k+1)^{N_1+\cdots+N_{j-1}+M-j+1}}$-coloring of $\{N_1+\cdots+N_{j-1}+1, \ldots, N_1+\cdots+N_j\}$, there is a monochromatic member of $\mathcal{P}$. Continue until $N_1, \ldots, N_M$ have been chosen.

By choice of N_M, there exists $\alpha_M \subset \{N_1+\cdots+N_{M-1}+1, \ldots, N_1+\cdots+N_M\}$ with $\alpha_M \in \mathcal{P}$ such that for all words w of length $N_1+\ldots+N_{M-1}$ and all $t_M \in \{0, 1, \ldots, k\}$, the function $\alpha_M \to \{1, \ldots, r\}$ defined by $i \to c(wv_{i,t_M})$ takes on a constant value depending only on t_M and w; here v_{i,t_M} is the word of length N_M having all entries "0" except for a single entry "t_M", located so as to occur in the ith place of wv_{i,t_M}.

Having chosen $\alpha_M, \alpha_{M-1}, \ldots, \alpha_{j+1}$, select (by choice of N_j) $\alpha_j \subset \{N_1+\cdots+N_{j-1}+1, \ldots, N_1+\cdots+N_j\}$ with $\alpha_j \in \mathcal{P}$ such that for all words w of length $N_1+\ldots+N_{j-1}$ and all $t_j, t_{j+1}, \ldots, t_M \in \{0, 1, \ldots, k\}$, the function $\alpha_j \to \{1, \ldots, r\}$ defined by $i \to c(wv_{i,t_j,t_{j+1},\ldots,t_M})$ takes on a constant value depending only on the t_as and w; here $v_{i,t_j,t_{j+1},\ldots,t_M}$ is any word of length $N_j+\cdots+N_M$ having all entries "0" except for entries "t_a", $j \le a \le M$, located so as to occur in places $i, i_{j+1}, \ldots, i_M$, respectively, of $wv_{i,t_j,t_{j+1},\ldots,t_M \in \{0,1,\ldots,k\}}$, where $i_a \in \alpha_a$ are arbitrary, $j < a \le M$.

By prior construction, $c(wv_{i,t_j,t_{j+1},\ldots,t_M})$ cannot depend on the i_as. The current construction shows it does not depend on i, either. Accordingly, once α_1 has been chosen, we are done. $\square$

1.7. Proof of Theorem 1.5. Let r, k and $\mathcal{P}$ be given. By the Hales-Jewett theorem there exists $M \in \mathbf{N}$ such that if $\{0, 1, \ldots, k\}^M$ is r-colored, there is a variable word $w(x)$ of length M such that $\{w(t) : 0 \le t \le k\}$ is monochromatic. For this M, let N be as in Lemma 1.6. Let now an r-coloring $c : \{0, 1, \ldots, k\}^N \to \{1, 2, \ldots, r\}$ be given, and let $\alpha_1, \ldots, \alpha_M$ be as guaranteed by Lemma 1.6.

We induce an r-coloring $d : \{0, 1, \ldots, k\}^M \to \{1, 2, \ldots, r\}$ as follows: for a word $t_1 t_2 \cdots t_M$ of length M, pick i_a in α_a, $1 \le a \le M$, and put $d(t_1 t_2 \cdots t_M) = c(v_{t_1,\ldots,t_M})$, where $v_{t_1,\ldots,t_M}$ is any word of length N having all zero entries except entries of t_a at places i_a, respectively, where $i_a \in \alpha_a$, $1 \le a \le M$. (That d is well-defined is the content of Lemma 1.6.)

By choice of M, there is a variable word $v(x) = v_1 v_2 \cdots v_M$ such that $t \to d\big(v(t)\big)$ takes on a constant value d. Let $b_1 < b_2 < \cdots < b_l$ be those indices b for which $v_b = x$. Put $\beta_j = \alpha_{b_j}$, $1 \le j \le l$, and let $w = u_1 u_2 \cdots u_N$, where for each b with $v_b \ne x$, there is exactly one $i \in \alpha_b$ for which $u_i = v_b$, and all u_i not so defined are 0.

If now $j_i \in \beta_i, 1 \le i \le l$, and $0 \le t \le k$, $c\big(w^{\{j_1,\ldots,j_l\}}(t)\big) = d\big(v(t)\big) = d$. $\square$

1.8. Corollary. Let $k, r \in \mathbf{N}$ and suppose that $\mathcal{P}$ is a (*not necessarily dilation invariant*) partition regular family of finite subsets of $\mathbf{N}$. There exists $M \in \mathbf{N}$ having the property that for every r-coloring of $\{1, 2, \ldots, M\}$, there exist $l, b \in \mathbf{N}$, and $\beta_i \in \mathcal{P}$, $1 \le i \le l$ with $\beta_1 < \beta_2 < \cdots < \beta_l$, such that

$$\big\{b(j_1 j_2 \cdots j_l)^t : j_i \in \beta_i, 1 \le i \le l, 0 \le t \le k\big\}$$

is monochromatic.

Proof. Choose N as in Theorem 1.5 and put $M = \prod_{i=1}^{N} i_k$. Let $d : \{1, 2, \ldots, M\} \to \{1, \ldots, r\}$ be a coloring. Define $\varphi : \{0, 1, \ldots, k\}^{\{1,\ldots,N\}} \to \{1, \ldots, M\}$ by $\varphi(w_1 \cdots w_N) = \prod_{i=1}^{N} i^{w_i}$ and $c : \{0, 1, \ldots, k\}^{\{1,\ldots,N\}} \to \{1, \ldots, r\}$ by $c(w) = d\big(\varphi(w)\big)$. Let $l, \beta_1, \ldots, \beta_l$ and $u_1 u_2 \cdots u_N$ be as guaranteed by Theorem 1.5 for this coloring c. Letting $b = \varphi(u_1 \cdots u_N)$ we get the desired result. $\square$

2. How to Beiglböck a Carlson-Simpson

Although we did not require our partition regular family $\mathcal{P}$ in the previous section to be shift invariant, in many practical applications it will be (for example, $\mathcal{P}$ might be the set of arithmetic progressions of a fixed length). In case $\mathcal{P}$ is shift invariant, the family of configurations guaranteed by Theorem 1.5 is shift invariant as well (in the semigroup of words), and it becomes possible to prove, along well established lines, an infinitary version of the result.

2.1. The standard infinitary instance of the Hales-Jewett theorem is due to T. Carlson and S. Simpson [CS]. Fix k and let $\mathcal{W}$ be the family of words on the alphabet $\{0, 1, \ldots, k\}$. $\mathcal{W}$ is a semigroup under concatenation. A weak form of the Carlson-Simpson theorem states that for any finite coloring of $\mathcal{W}$, there is a sequence of variable words $\big(w_i(x)\big)$ such that

$$\Big\{w_1(t_1)w_2(t_2)\cdots w_s(t_s) : s \in \mathbf{N}, t_i \in \{0, 1, \ldots, k\}, 1 \le i \le s\Big\}$$

is monochromatic. (The full strength of the theorem requires that the leftmost letter of w_i, $i > 1$, be "x".)

2.2. A subset $A \subset \mathcal{W}$ is *right syndetic* if there is a finite set $F \subset \mathcal{W}$ such that $F^{-1}A = \{y : fy \in A \text{ for some } f \in F\} = \mathcal{W}$. A set $T \subset \mathcal{W}$ is *right thick* if for every finite set $F \subset \mathcal{W}$, there is some $w \in \mathcal{W}$ such that $Fw \subset T$. One may check that a set is right syndetic if and only if it meets every right thick set, and a set is right thick if and only if it meets every right syndetic set. Finally, a set $P \subset \mathcal{W}$ is *right piecewise syndetic* if there is a finite set $F \subset \mathcal{W}$ such that $F^{-1}A$ is right thick. (Warning: some authors use "left" in place of "right" in the above definitions, including the second author in the past and, it seems likely, in the future.)

2.3. Lemma. Let $\mathcal{R}$ be a shift-invariant partition regular family of finite subsets of $\mathcal{W}$. Then any right piecewise syndetic set contains a member of $\mathcal{R}$.

Proof. Let A be right piecewise syndetic and choose a finite set F such that $F^{-1}A$ is right thick. Let $G_1, G_2, \ldots$ be an increasing sequence of finite sets exhausting $\mathcal{W}$. For each i, pick w_i such that $G_i w_i \subset F^{-1}A$. Let $c_i : G_i \to F$ be a function having the property that $g w_i \in c_i(g)^{-1}A$ for all $g \in G_i$. Let c be a weak limit point of the sequence (c_i). By partition regularity of $\mathcal{R}$, there is a set $R \in \mathcal{R}$ on which c is constant. For some i, therefore, c_i is constant on R; say $c_i(g) = f$ for all $g \in R$. Then $Rw_i \subset f^{-1}A$, so that $fRw_i \subset A$. $\square$

2.4. Let $\beta\mathcal{W}$ be the Stone-Čech compactification of $\mathcal{W}$. We take the points of $\beta\mathcal{W}$ to be ultrafilters on $\mathcal{W}$, and extend the semigroup operation to $\beta\mathcal{W}$ by the rule $A \in pq$ if and only if $\{w : w^{-1}A \in q\} \in p$. As is well known, $\beta\mathcal{W}$ has a smallest ideal, this ideal

contains idempotents (so-called *minimal* idempotents), and any member of any minimal idempotent is right piecewise syndetic. See [HS, Chapter 4] for more details.

2.5. Theorem. Fix $k \in \mathbf{N}$ and let $\mathcal{P}$ be a shift invariant, partition regular family of finite subsets of $\mathbf{N}$. For any finite coloring of $\mathcal{W} = \bigcup_{N=1}^{\infty}\{0,1,\ldots,k\}^N$, there exist sequences of natural numbers (l_i) and (N_i), sets $\beta_i^{(j)} \in \mathcal{P}$, $j \in \mathbf{N}$, $1 \le i \le l_j$, with $\beta_1^{(j)} < \beta_2^{(j)} < \cdots < \beta_{l_j}^{(j)} < \{N_j+1\}$, and a sequence of words (w_i) in $\mathcal{W}$ with len $w_i = N_i$, w_j having 0 at indices in $\bigcup_{i=1}^{l_j} \beta_i^{(j)}$, such that the following set is monochromatic:

$$\Big\{ w_{a_1}^{\{j_1^{(1)},\ldots,j_{l_{a_1}}^{(1)}\}}(t_1) w_{a_2}^{\{j_1^{(2)},\ldots,j_{l_{a_2}}^{(2)}\}}(t_2) \cdots w_{a_s}^{\{j_1^{(s)},\ldots,j_{l_{a_s}}^{(s)}\}}(t_s) : a_1 < a_2 < \cdots < a_s, \\ j_i^{(b)} \in \beta_i^{(b)}, 1 \le i \le l_{a_b}, t_b \in \{0,1,\ldots k\}, 1 \le b \le s \Big\}.$$

Proof. Let $\gamma : \mathcal{W} \to \{1,2,\ldots,r\}$ be a finite coloring. Select a minimal idempotent p and choose $j \in \{1,2,\ldots,r\}$ such that $B_1 = \gamma^{-1}(\{j\}) \in p$. Since p is idempotent, $\big(B_1 \cap \{w : w^{-1}B_1 \in p\}\big)$ is a member of p and hence piecewise right syndetic.

Let $\mathcal{R}$ be the family of subsets

$$\Big\{ w^{\{j_1,\ldots,j_l\}}(t) : j_i \in \beta_i, 1 \le i \le l, 0 \le t \le k \Big\},$$

where $\beta_i \in \mathcal{P}$, $1 \le i \le l$, with $\beta_1 < \beta_2 < \cdots < \beta_l < \{N+1\}$, and $w = u_1 u_2 \cdots u_N \in \{0,1,\ldots,k\}^N$ has the property that $u_i = 0$ for all $i \in \bigcup_{j=1}^{l} \beta_j$. $\mathcal{R}$ is clearly shift-invariant and is partition regular by Theorem 1.4. Therefore, by Lemma 2.2, we may select $l_1, N_1 \in \mathbf{N}$, $\beta_1^{(1)},\ldots,\beta_{l_1}^{(1)} \in \mathcal{P}$, with $\beta_1^{(1)} < \cdots < \beta_{l_1}^{(1)} < \{N_1+1\}$, a word w_1 of length N_1 that is 0 on $\bigcup_{j=1}^{l_1} \beta_j^{(1)}$, such that

$$S_1 = \Big\{ w_1^{\{j_1^{(1)},\ldots,j_{l_1}^{(1)}\}}(t_1) : j_i^{(1)} \in \beta_i^{(1)}, 1 \le i \le l_1, 0 \le t_1 \le k \Big\} \subset \big(B_1 \cap \{w : w^{-1}B_1 \in p\}\big)$$

Let now $B_2 = B_1 \cap \bigcap_{w \in S_1} w^{-1}B_1$. Then $B_2 \in p$, which implies that $\big(B_2 \cap \{w : w^{-1}B_2 \in p\}\big)$ is a member of p, and in particular right piecewise syndetic. Accordingly, we may select $l_2, N_2 \in \mathbf{N}$, $\beta_1^{(2)},\ldots,\beta_{l_2}^{(2)} \in \mathcal{P}$, with $\beta_1^{(2)} < \cdots < \beta_{l_2}^{(2)} < \{N_2+1\}$, a word w_2 of length N_2 that is 0 on $\bigcup_{j=1}^{l_2} \beta_j^{(2)}$, such that

$$S_2 = \Big\{ w_2^{\{j_1^{(2)},\ldots,j_{l_2}^{(2)}\}}(t_2) : j_i^{(2)} \in \beta_i^{(2)}, 1 \le i \le l_2, 0 \le t_2 \le k \Big\} \subset \big(B_2 \cap \{w : w^{-1}B_2 \in p\}\big).$$

One may now check that, for example,

$$w_1^{\{j_1^{(1)},\ldots,j_{l_1}^{(1)}\}}(t_1) w_2^{\{j_1^{(2)},\ldots,j_{l_2}^{(2)}\}}(t_2) \in B_1$$

for all appropriate choices of the parameters involved. As our proof follows a well-established paradigm, it should be clear by now how to proceed. $\square$

3. An explanation revisited

We got to Theorem 2.5 via infinitary upgrades to Theorem 1.7, which in turn followed from Lemma 1.6 and the Hales-Jewett theorem. We now go back and investigate the prospect of an infinitary version of this lemma.

3.1. Definition. For $k \in \mathbf{N}$, let $\mathcal{L}$ be the set of *located words* on $\{0, 1, \ldots, k\}$, i.e. functions $f : \alpha \to \{0, 1, \ldots, k\}$, where α is a finite subset (possibly empty) of $\mathbf{N}$. We take members of $\mathcal{L}$ to be sets of ordered pairs. Seen in this light, $\mathcal{L}$ is not quite a semigroup under union (the union of two functions may fail to be a function). In order to facilitate the proof to come, it will be convenient for us to pick some $\omega \notin \mathcal{L}$, let $\mathcal{L}_\omega = \{\omega\} \cup \mathcal{L}$ and interpret $\cup$ as a binary operation on $\mathcal{L}_\omega$ in the following way:

a. for f, $g \in \mathcal{L}$, $f \cup g$ is the union of f and g provided f and g have disjoint domains, and $f \cup g = \omega$ otherwise.

b. for $f \in \mathcal{L}$, $f \cup \omega = \omega \cup f = \omega \cup \omega = \omega$.

Then $(\mathcal{L}_\omega, \cup)$ is a semigroup.

3.2. Beiglböck's original formulation of his theorem follows.

Theorem. Let $k \in \mathbf{N}$ and let $\mathcal{P}$ be a partition regular family of finite subsets of $\mathbf{N}$. For any finite coloring of $\mathcal{L}$, there is a monochromatic configuration of the form

$$\big\{f \cup (\gamma \cup \{t\}) \times \{j\} : j \in \{0, 1, \ldots, k\}, t \in P\big\},$$

where $P \in \mathcal{P}$, $f \in \mathcal{L}$ and $\gamma \subset \mathbf{N}$ is finite with $\mathrm{Dom} f \cap (\gamma \cup P) = \emptyset$.

3.3. Here now is our infinitary version of Lemma 1.6, cast in this language.

Theorem. Let $r, k, M \in \mathbf{N}$, and suppose $\mathcal{P}$ is a partition-regular family of finite subsets of $\mathbf{N}$. For any r-coloring $x : \mathcal{L} \to \{1, 2, \ldots, r\}$, there exist $\alpha_1, \alpha_2, \ldots \in \mathcal{P}$ with $\alpha_1 < \alpha_2 < \cdots$, and an r-coloring c of words on $\{0, 1, \ldots, k\}$ having length at most M, such that for any $1 \le z \le M$, $1 \le m_1 < m_2 < \cdots < m_z$, $n_i \in \alpha_{m_i}$ and $j_i \in \{0, 1, \ldots, k\}$, $1 \le i \le z$, one has

$$x\big(\{(n_1, j_1), (n_2, j_2), \ldots, (n_z, j_z)\}\big) = c(j_1 j_2 \cdots j_z).$$

Proof. Let $X = \{1, 2, \ldots, r\}^{\mathcal{L}_\omega}$ with the product topology and set $\Omega = X^X$, again with the product topology. For $g \in \mathcal{L}$, define $T_g \in \Omega$ by $T_g q(f) = q(f \cup g)$ for $q \in X$ and $f \in \mathcal{L}_\omega$. This embeds $\mathcal{L}$ in Ω. Put $E = \overline{\{T_g : g \in \mathcal{L}\}} \subset X^X$.

We now embed $\mathbf{N}$ in E^{k+1} by the map

$$\theta(n) = \begin{pmatrix} T_{\{(n,0)\}} \\ T_{\{(n,1)\}} \\ \vdots \\ T_{\{(n,k)\}} \end{pmatrix}$$

and let $\overline{\theta} : \beta\mathbf{N} \to E^{k+1}$ be the (unique) extension of θ to $\beta\mathbf{N}$. Let p be any ultrafilter on $\mathbf{N}$ having the property that any member of p contains a member of $\mathcal{P}$ (cf. [HS, Theorem 3.11 (b)). Put

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_k \end{pmatrix} = \overline{\theta}(p).$$

Let $x \in X$ be an r-coloring of $\mathcal{L}$. Now let A_1 be the set of $n \in \mathbf{N}$ such that for all choices $0 \leq s < M$ and $\{j, i_1, \ldots, i_s\} \subset \{0, 1, \ldots, k\}$, one has

$$\phi_j \phi_{i_1} \phi_{i_2} \cdots \phi_{i_s} x(\emptyset) = \phi_{i_1} \phi_{i_2} \cdots \phi_{i_s} x\big(\{(n, j)\}\big).$$

Since $A_1 \in p$, we can choose $\alpha_1 \in \mathcal{P}$ with $\alpha_1 \subset A_1$. Note in particular that for $n \in \alpha_1$, $x\big(\{(n, j)\}\big) = \phi_j x(\emptyset)$, which does not depend on n.

For induction, suppose one has chosen $\alpha_1, \ldots, \alpha_{t-1} \in \mathcal{P}$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_{t-1}$ and that for $1 \leq z \leq M$, $1 \leq m_1 < m_2 < \cdots < m_z < t$, $n_i \in \alpha_{m_i}$ and $j_i \in \{0, 1, \ldots, k\}$, $1 \leq i \leq z$, one has

$$x\big(\{(n_1, j_1), (n_2, j_2), \ldots, (n_z, j_z)\}\big) = \phi_{j_1} \cdots \phi_{j_{z-1}} \phi_{j_z} x(\emptyset).$$

Let A_t be the set of $n \in \mathbf{N}$ with $n > \max \alpha_{t-1}$ such that for all choices $0 \leq s < M$, $\{j, i_1, \ldots, i_s\} \subset \{0, 1, \ldots, k\}$, $1 \leq m_1 < m_2 < \cdots < m_{z-1} < t$, $n_i \in \alpha_{m_i}$ and $j_i \in \{0, 1, \ldots, k\}$, $1 \leq i < z$, one has

$$\begin{aligned} &\phi_j \phi_{i_1} \phi_{i_2} \cdots \phi_{i_s} x\big(\{(n_1, j_1), (n_2, j_2), \ldots, (n_{z-1}, j_{z-1})\}\big) \\ =&\phi_{i_1} \phi_{i_2} \cdots \phi_{i_s} x\big(\{(n_1, j_1), (n_2, j_2), \ldots, (n_{z-1}, j_{z-1}), (n, j)\}\big). \end{aligned}$$

Since $A_t \in p$, we can choose $\alpha_t \in \mathcal{P}$ with $\alpha_t \subset A_t$.

One now routinely checks that for $1 \leq z \leq M$, $1 \leq m_1 < m_2 < \cdots < m_z = t$, $n_i \in \alpha_{m_i}$ and $j_i \in \{0, 1, \ldots, k\}$, $1 \leq i \leq z$, one has

$$\begin{aligned} x\big(\{(n_1, j_1), (n_2, j_2), \ldots, (n_z, j_z)\}\big) &= \phi_{j_z} x\big(\{(n_1, j_1), (n_2, j_2), \ldots (n_{z-1}, j_{z-1})\}\big) \\ &= \phi_{j_{z-1}} \phi_{j_z} x\big(\{(n_1, j_1), (n_2, j_2), \ldots, (n_{z-2}, j_{z-2})\}\big) \\ &\vdots \\ &= \phi_{j_1} \cdots \phi_{j_{z-1}} \phi_{j_z} x(\emptyset), \end{aligned}$$

which does not depend on n_i, $1 \leq i \leq z$. $\square$

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