# An Infinitary Polynomial Hales-Jewett Theorem

**Abstract:** A joint extension of H. Furstenberg's Central Sets Theorem, the Hales-Jewett coloring theorem and the polynomial van der Waerden theorem of V. Bergelson and A. Leibman is obtained by an elaboration on Furstenberg and Y. Katznelson's approach to infinitary Ramsey theory via the enveloping semigroup.

## 0. Introduction

Van der Waerden's theorem ([vdW]) states that for any finite partition of the natural numbers $\mathbf{N}$, at least one of the cells contains arbitrarily long arithmetic progressions. There are at least three senses in which van der Waerden's theorem has been extended (in [HJ], [F] and [BL1]). Moreover, joint extensions have been given for any two of these three directions considered together (in [CS], [BL2] and [M]). The purpose of this paper is to provide an extension of van der Waerden's theorem in all three directions simultaneously. Specifically, we produce a "polynomial version" of an infinitary Hale-Jewett type theorem due to T. Carlson and S. Simpson.

The first direction of extension we consider was originally provided by V. Bergelson and A. Leibman, who in [BL1] considered "polynomial progressions" instead of arithmetic progressions. A special case of their result states that if $p_1(x), \cdots, p_k(x) \in \mathbf{Z}[x]$ with $p_i(0) = 0$, $1 \leq i \leq k$, then if $\mathbf{Z} = \bigcup_{i=1}^{r} C_i$ is a finite partition then for some $i$, $1 \leq i \leq r$, some $a \in \mathbf{Z}$, and some $n > 0$, $\{a + p_1(n), a + p_2(n), \cdots, a + p_k(n)\} \subset C_i$.

The second direction of extension is exemplified by the classical Hales-Jewett coloring theorem. For $M, k \in \mathbf{N}$, let $\mathcal{W}_M^{(k)}$ denote the set of all words of length $M$ using the "letters" $\{0, 1, \cdots, k-1\}$. Write also $\mathcal{W}^{(k)} = \bigcup_{M=1}^{\infty} \mathcal{W}_M^{(k)}$. A *variable word* of length $M$ is a word $w(x)$ on the letters $\{0, 1, \cdots, k-1, x\}$, where the "variable" $x$ appears at least once. A length $M$ variable word $w(x)$ gives rise in a natural way (via simple replacement) to a function $w : \{0, 1, \cdots, k-1\} \to \mathcal{W}_M^{(k)}$. The Hales-Jewett theorem states that for any $k, r \in \mathbf{N}$, there exists $M = M(k, r)$ such that if $\mathcal{W}_M^{(k)} = \bigcup_{i=1}^{r} C_i$ then some cell $C_i$ contains a configuration of the form $\{w(i) : i = 0, 1, \cdots, k-1\}$, where $w(x)$ is a length $M$ variable word. (To see that this in fact extends van der Waerden's theorem, identify the word $i_1 i_2 \cdots i_M$ with the natural number $1 + \sum_{j=1}^{M} i_j k^{j-1}$ and check that the resulting configuration corresponds to an arithmetic progression of length $k$.)

Although it is not completely clear from the above discussion, the Hales-Jewett theorem has a strictly set-theoretic formulation that may be "polynomialized". Indeed, if one replaces numbers by sets in an appropriate Cartesian product space, then replaces addition and multiplication by set-theoretic union and product, respectively, one may develop a language in which it is possible to formulate a "polynomial Hales-Jewett theorem". Such a result was proved, again by Bergelson and Leibman, in [BL2].

The set-up is as follows. Let $l \in \mathbf{N}$ be fixed for the time being. A *set-monomial* (over $\mathbf{N}^l$) in the variable $X$ is an expression $m(X) = S_1 \times S_2 \times \cdots \times S_l$, where for each $i$, $1 \leq i \leq l$, $S_i$ is either the symbol $X$ or a non-empty singleton subset of $\mathbf{N}$. The *degree* of the monomial is the number of times the symbol $X$ appears in the list $S_1, \cdots, S_l$. For example, taking $l = 3$, $m(X) = \{5\} \times X \times X$ is a set-monomial of degree 2, while $m(X) =$

$X \times \{17\} \times \{2\}$ is a set-monomial of degree 1. A *set-polynomial* is an expression of the form $p(X) = m_1(X) \cup m_2(X) \cup \cdots \cup m_k(X)$, where $m_1(X), \cdots, m_k(X)$ are set-monomials. The degree of a set-polynomial is the largest degree of its set-monomial "summands", and its *constant term* consists of the "sum" of those $m_i$ that are constant, i.e. of degree zero.

Letting $\mathcal{F}(S)$ denote the family of non-empty finite subsets of a set $S$, any non-empty set polynomial $p(A)$ determines a function from $\mathcal{F}(\mathbf{N})$ to $\mathcal{F}(\mathbf{N}^l)$ in the obvious way (interpreting the symbol $\times$ as Cartesian product and the symbol $\cup$ as union). The following is a consequence of [BL2, Theorem 3.5].

**Theorem 0.1** Let $l \in \mathbf{N}$ and let $p_1(X), \cdots, p_k(X)$ be set-polynomials over $\mathbf{N}^l$ whose constant terms are empty. Let $H \subset \mathbf{N}$ be any finite set and let $r \in \mathbf{N}$. There exists a finite set $M \subset \mathbf{N}$, with $M \cap H = \emptyset$, having the property that if $\mathcal{F}(\mathbf{N}^l) = \bigcup_{i=1}^r C_i$ then there exists $j$, $1 \leq j \leq r$, some non-empty $N \subset M$, and some $A \subset \bigcup_{i=1}^k p_i(M)$ such that $A \cap p_i(N) = \emptyset$, $1 \leq i \leq k$, and

$$\{A \cup p_1(N), A \cup p_2(N), \cdots, A \cup p_k(N)\} \subset C_j.$$

To pave the way for the third type of extension we consider, let us introduce some notation and definitions. Let $\mathcal{F} = \mathcal{F}(\mathbf{N})$ denote the family of finite, non-empty subsets of $\mathbf{N}$. For $\alpha, \beta \in \mathcal{F}$, write $\alpha < \beta$ if for all $n \in \alpha$ and all $m \in \beta$, $n < m$. An *IP-ring* consists of all non-empty finite unions taken from an increasing sequence $(\alpha_i)_{i=1}^{\infty} \subset \mathcal{F}$ (notice that $\mathcal{F}$ itself is an IP-ring, corresponding to the sequence $\alpha_i = \{i\}$). If $\mathcal{F}^{(1)}$ is an IP-ring, a sequence $(x_\alpha)_{\alpha \in \mathcal{F}^{(1)}}$ in an additive abelian semigroup will be called an *IP-set* if $x_{\alpha \cup \beta} = x_\alpha + x_\beta$ whenever $\alpha \cap \beta = \emptyset$.

Now, the final manner we consider in which van der Waerden's theorem may be extended is by seeking infinite configurations. The prototypical model for this genre of "infinitary Ramsey theory" is Hindman's theorem ([H]), which states that for any finite partition of an IP-set, some cell contains an sub-IP-set (hence the infinite monochromatic configuration arrived at has the identical structure of the object originally colored). The first really satisfactory example of an infinitary van der Waeden-type theorem is Furstenberg's so-called "central sets theorem" ([F, Proposition 8.21]), a special case of which guarantees a monochromatic "IP-set of arithmetic progressions" for any finite coloring of $\mathbf{N}$.

The simultaneous extension in the infinitary and set-theoretic directions is due to Carlson and Simpson ([CS]). We give now a (somewhat weakened) version of their theorem. (The full strength of their result implies that all but the first variable word $w_i(x)$ appearing in the formulation below may be chosen with $x$ as the left-most letter.)

**Theorem 0.2** Let $k, r \in \mathbf{N}$. If $\mathcal{W}^{(k)} = \bigcup_{i=1}^r C_i$ then there exists $j$, with $1 \leq j \leq r$, and a sequence of variable words $\{w_i(x)\}_{i=1}^{\infty}$ such that for all $M \in \mathbf{N}$ and all choices $i_t \in \{0, 1, \cdots, k-1\}$, $1 \leq t \leq M$, $w_1(i_1) w_2(i_2) \cdots w_M(i_M) \in C_j$.

Notice that, as is the case with Hindman's theorem, the substructure one obtains in one cell has the same form as the original structure (in this case, via the correspondence $i_1 i_2 \cdots i_M \leftrightarrow w_i(i_1) w_2(i_2) \cdots w_M(i_M)$).

Finally, an infinitary polynomial van der Waerden theorem was obtained in [M].

**Theorem 0.3** Let $\mathbf{N} = \bigcup_{i=1}^{r} C_i$, and suppose $(n_\alpha)_{\alpha\in\mathcal{F}}$ is an IP-set in $\mathbf{Z}$. Given polynomials $p_1(x), \cdots, p_k(x) \in \mathbf{Z}[x]$ with $p_i(0) = 0$, $1 \le i \le k$, there exists $j$, with $1 \le j \le r$, an IP-ring $\mathcal{F}^{(1)}$, and an IP-set $(a_\alpha)_{\alpha\in\mathcal{F}^{(1)}}$ such that for all $\alpha \in \mathcal{F}^{(1)}$,

$$\{a_\alpha + p_1(n_\alpha), a_\alpha + p_2(n_\alpha), \cdots, a_\alpha + p_k(n_\alpha)\} \subset C_j.$$

Our main result, Theorem 2.3, is an extension of van der Waerden's theorem in all three of the aforementioned senses. Moreover, it has the aesthetic property already observed in Hindman's theorem and the Carlson-Simpson theorem. Namely, the monochromatic configuration obtained is isomorphic to that which is initially colored.

The structure of the paper is as follows. In Section 1, we prove in detail a special case of a "set theoretic" formulation of our main theorem in which, for ease of notation, we limit the number of dimensions to 2. Next, we develop a "matrix terminology" then cast the main theorem in that. In Section 2, we expand the matrix terminology, and formulate our main theorem (Theorem 2.3) for any finite number of dimensions. Finally, we demonstrate that our result is indeed an extension of Theorem 0.2 by deriving Theorem 0.2 from it.

## 1. A quadratic infinitary Hales-Jewett type theorem.

A semigroup $S$ is a *compact left topological semigroup* if it is endowed with a topology with respect to which it is a compact Hausdorff space and with respect to which the map $t \to ts$ is continuous for all $s \in S$. (Note: some authors call this *right* topological.)

**Proposition 1.1.** (see [E]) Any compact left topological semigroup $S$ possesses an idempotent.

**Proof.** Let $\mathcal{M}$ denote the family of non-empty closed subsets $P \subset S$ for which $P^2 \subset P$. By Zorn's Lemma, $\mathcal{M}$ contains a minimal element $P$ with respect to inclusion. Let $p \in P$. Then $Pp \subset P$ is compact (being the continuous image of a compact set), non-empty, and moreover $(Pp)^2 \subset P$, hence $Pp = P$. In particular the set $Q = \{q \in P : qp = p\} \subset P$ is non-empty and, being the continuous inverse image of a singleton, closed. Furthermore $Q^2 \subset Q$, so that $Q = P$. That is, $qp = p$ for all $q \in P$. In particular, $p^2 = p$.

□

Let $S$ be a compact left topological semigroup and let $J \subset S$ be non-empty and closed. If $SJ \subset J$ then $J$ is said to be a *left ideal*. Any left ideal, itself being a compact left topological semigroup, contains an idempotent by Proposition 1.3. If $J$ is a left ideal of $S$ that is minimal among left ideals with respect to inclusion, then we call $J$ a *minimal left ideal*, and any idempotent $\theta \in J$ is called a *minimal idempotent*. By Zorn's Lemma every compact left topological semigroup contains a minimal left ideal and hence a minimal idempotent.

Let $\mathcal{G}_1 = \mathcal{G} = \mathcal{F}(\mathbf{N} \times \mathbf{N})$ be the set of all non-empty finite subsets of $\mathbf{N} \times \mathbf{N}$. $\mathcal{G}$ is an abelian semigroup under $\cup$ (that, is, union). For $N \ge 2$, let $\mathcal{G}_N = \mathcal{F}\big(\mathbf{N}^2 \setminus \{1, \cdots, N-1\}^2\big)$ be the family of elements in $\mathcal{G}$ that are disjoint from $\{1, \cdots, N-1\}^2$. Let $\mathcal{G}_0 = \mathcal{G} \cup \{\emptyset\}$.

We put $X = \{0, 1\}^{\mathcal{G}_0}$. Then $X$ (with the product topology) is compact and metrizable. With respect to the product topology, $X^X$ with composition of functions as the operation forms a compact left topological semigroup. We embed $\mathcal{G}_0$ in $X^X$ by putting $T_E\gamma(A) = \gamma(A \cup E)$ for $\gamma \in X$ and $A, E \in \mathcal{G}_0$. Finally we let

$$S = \bigcap_{N=1}^{\infty} \overline{\{T_E : E \in \mathcal{G}_N\}}. \tag{1.1}$$

Each of the members of this intersection forms a semigroup (see (2.3) in [FK] for details). Hence $S$ is a compact left topological semigroup as well. Furthermore, this is a decreasing intersection of closed sets, hence $S$ is non-empty by compactness.

Working in the semigroup $S$ as defined in (1.1) (rather than in simply $\overline{\{T_E : E \in \mathcal{G}_0\}}$) is one way of dealing with the non-cancellativity of $\mathcal{G}_0$. The idea is, knowing what $E$ and $E \cup A$ are, we can only recover what $A$ is if we know something else; specifically, if we can assume that $(E \cap A) = \emptyset$, then we will know that $A = (E \cup A) \setminus A$. Thus $A$ is "preserved" by (and only by) "disjoint shifts". Extending this to finite configurations, a configuration $\{A_1, \cdots, A_k\} \subset \mathcal{G}_0$ is in some sense "preserved" if one shifts by a set $E$ which is disjoint from all of the $A_i$'s. That is, if one knows $E$ and $\{E \cup A_1, \cdots, E \cup A_k\}$, and that $E$ is disjoint from the $A_i$'s, one may recover $\{A_1, \cdots, A_k\}$. The classes of configurations we deal with in this section are, indeed, closed under these disjoint shifts, but badly non-closed under arbitrary shifts. By living in the semigroup $S$ defined above, we can, given $\phi \in S$ and any configuration $\{A_1, \cdots, A_k\} \subset \mathcal{G}_0$, always approximate $\phi$ by some $T_E$, where $E$ is disjoint from each $A_i$. This is important in the sequel.

An alternative approach to this issue arises by only allowing unions between disjoint sets. One advantage of doing this is that the operation becomes cancellative. A disadvantage is that the object $(\mathcal{G}, \cup)$ is no longer a semigroup, but a "partial" semigroup. This approach to infinitary Ramsey theoretic matters originates in [BBH], and is developed further in [HM]. We shall not, however, employ it here.

A subset $E$ of an abelian semigroup $S$ is *syndetic* if there exists a finite set $F \subset S$ such that $S = \bigcup_{x \in F} x^{-1}E$. (Here $y \in x^{-1}E$ if $xy \in E$.) $E$ is *piecewise syndetic* if there exists a syndetic set $B$ such that any finite subset of $B$ can be shifted into $E$. In the present context, these well-known ideas are not especially useful. Instead, we introduce two related notions.

**Definition 1.2** Suppose $\mathcal{E} \subset \mathcal{G}$. $\mathcal{E}$ is said to be *strongly syndetic* if for every $M \in \mathbf{N}$, there exists $N \in \mathbf{N}$ such that for all $E \in \mathcal{G}_{N+1}$, there exists $C \subset \{1, \cdots N\}^2 \setminus \{1, \cdots, M\}^2$ such that $E \cup C \in \mathcal{E}$. $\mathcal{E}$ is said to be *strongly piecewise syndetic* if there exists a strongly syndetic set $\mathcal{B} \subset \mathcal{G}$ such that for every finite family $\mathcal{H} \subset \mathcal{B}$ and every $N \in \mathbf{N}$ there exists $E \in \mathcal{G}_{N+1}$ such that $(E \cup F) \in \mathcal{E}$ for every $F \in \mathcal{H}$.

Suppose now that $k \in \mathbf{N}$ and $p_1(B), \cdots, p_k(B)$ are set-polynomials over $\mathbf{N}^2$ having empty constant term. Let $\mathcal{A}$ be the family of configurations

$$\mathcal{A} = \big\{\{A \cup p_1(B), A \cup p_2(B), \cdots, A \cup p_k(B)\} : \ A \in \mathcal{G}, B \in \mathcal{F}, \big(A \cap p_i(B)\big) = \emptyset, \ 1 \le i \le k\big\}.$$

Now, according to Theorem 0.2, for any $M \geq 0$ and for any finite coloring of $\mathcal{G}_{M+1}$ there exists a monochromatic member of $\mathcal{A}$. In particular, for every finite coloring of $\mathcal{G}_0$ there exists a monochromatic configuration $\{A \cup p_1(B), A \cup p_2(B), \cdots, A \cup p_k(B)\}$ with

$\big(A \cup p_i(B)\big) \in \mathcal{G}_{M+1}$, $1 \leq i \leq k$. We will call such families (families which are partition regular in $\mathcal{G}_{M+1}$ for all $M$) of configurations *strongly partition regular.*

**Proposition 1.3** Let $\mathcal{E} \subset \mathcal{G}$, let $M \in \mathbf{N}$ and let $\mathcal{A}$ be any strongly partition regular family of configurations closed under disjoint shifts.

(a) If $\mathcal{E}$ is strongly syndetic then $\mathcal{E}$ contains a member of $\mathcal{A}$ all of whose elements belong to $\mathcal{G}_{M+1}$.

(b) If $\mathcal{E}$ is strongly piecewise syndetic then $\mathcal{E}$ contains a member of $\mathcal{A}$ all of whose elements belong to $\mathcal{G}_{M+1}$.

**Proof.** (a) Let $N \in \mathbf{N}$ be large enough that for every $E \in \mathcal{G}_{N+1}$ there exists $A \subset \big(\{1,\cdots,N\}^2 \setminus \{1,\cdots,M\}^2\big)$ such that $(E \cup A) \in \mathcal{E}$. Indeed, finitely partition $\mathcal{G}_{N+1}$ by assigning $E$ to a cell according to which $A$ accomplishes this (there are finitely many choices for $A$). For this coloring, there exists a monochromatic configuration $\mathcal{H} \in \mathcal{A}$. Monochromaticity implies that for some fixed $A \subset \{1,\cdots,N\}^2 \setminus \{1,\cdots,M\}^2$, $A \cup \mathcal{H} = \{A \cup H : H \in \mathcal{H}\} \subset \mathcal{E}$. But $\mathcal{A}$ is closed under disjoint shifts, so $(A \cup \mathcal{H}) \in \mathcal{A}$. Furthermore, $A \in \mathcal{G}_{M+1}$ and $\mathcal{H} \subset \mathcal{G}_{N+1}$, so we are done.

(b) If $\mathcal{E}$ is strongly piecewise syndetic then there exists a strongly syndetic $\mathcal{B}$ such that every finite family contained in $\mathcal{B}$ can be moved (shifting by an element arbitrarily far out) into $\mathcal{E}$. Therefore this part follows at once from part (a).

□

The following fact will look very familiar to aficionados of the ultrafilter approach to Ramsey theory (cf. [HS, Theorem 4.40]).

**Proposition 1.4** Let $J$ be a minimal ideal in $S$ and let $\theta \in J$. Let $t \in \mathbf{N}$, $\gamma_1, \cdots, \gamma_t \in X$ and $l \geq 0$. Then:

(a) $B_l = \{E : \theta\gamma_j(E \cup A) = \theta\gamma_j(A),\ 1 \leq j \leq t,\ A \subset \{1,2,\cdots,l\}^2\}$ is strongly syndetic.

(b) $P_l = \{E : \gamma_j(E \cup A) = \theta\gamma_j(A),\ 1 \leq j \leq t,\ A \subset \{1,2,\cdots,l\}^2\}$ is strongly piecewise syndetic.

**Proof.** (a) Suppose $B_l$ is not strongly syndetic. Then there exists $M$ such that for all $N > M$ there exists a set $E_N \in \mathcal{G}_{N+1}$ such that for every $C \subset \{1,2,\cdots,N\}^2 \setminus \{1,2,\cdots,M\}^2$, $(E_N \cup C) \in B_l^c$. That is, for some $A \subset \{1,2,\cdots,l\}^2$ and $1 \leq j \leq t$,

$$T_{E_N}\theta\gamma_j(C \cup A) = \theta\gamma_j(E_N \cup C \cup A) \neq \theta\gamma_j(A).$$

Let $\phi$ be an accumulation point in $X^X$ of $\{T_{E_N} : N > M\}$. Then $\phi \in S$. Moreover, for every $C \in \mathcal{G}_{M+1}$, $\phi\theta\gamma_j(C \cup A) \neq \theta\gamma_j(A)$ for some $A \subset \{1,2,\cdots,l\}^2$ and some $j$ with $1 \leq j \leq t$.

Using the fact that $J$ is a minimal ideal, pick $\psi \in S$ such that $\psi\phi\theta = \theta$. Finally, choose $C \in \mathcal{G}_{M+1}$ such that $T_C$ is close enough to $\psi$ to ensure that

$$\phi\theta\gamma_j(C \cup A) = T_C\phi\theta\gamma_j(A) = \psi\phi\theta\gamma_j(A) = \theta\gamma_j(A),\ 1 \leq j \leq t,\ A \subset \{1,2,\cdots,l\}^2.$$

This establishes (a).

(b) Let $\mathcal{H} \subset B_l$ be a finite family and let $N \in \mathbf{N}$. Pick $E \in \mathcal{G}_{N+1}$ such that $T_E$ is close enough to $\theta$ to ensure that

$$\gamma_j(E \cup H \cup A) = T_E \gamma_j(H \cup A) = \theta \gamma_j(H \cup A) = \theta \gamma_j(A),$$
$$H \in \mathcal{H},\ 1 \le j \le t,\ A \subset \{1, 2, \cdots, l\}^2.$$

Then $(E \cup H) \in P_l$ for all $H \in \mathcal{H}$.

□

**Corollary 1.5** If $r \in \mathbf{N}$ and $\mathcal{G}_0 = \bigcup_{i=1}^r C_i$, then there exists $i$ with $1 \le i \le r$ such that $C_i$ is strongly piecewise syndetic.

**Proof.** Let $\theta \in S$ be an element of a minimal left ideal. Then

$$\theta \in \overline{\{T_E : E \in \mathcal{G}_0\}} = \bigcup_{i=1}^r \overline{\{T_E : E \in C_i\}},$$

so that for some $j$, $\theta \in \overline{\{T_E : E \in C_j\}}$. Therefore, there exists $E \in C_j$ such that $\theta 1_{C_j}(\emptyset) = T_E 1_{C_j}(\emptyset) = 1_{C_j}(E) = 1$. We now employ Proposition 1.4 with $l = 0$ and $t = 1$, taking $\gamma_1$ to be $1_{C_j}$. Hence the set $P = \{E : 1_{C_j}(E) = \theta 1_{C_j}(\emptyset)\}$ is strongly piecewise syndetic. But this set is simply $C_j$.

□

Finally, we have the following.

**Theorem 1.6** Let $J \subset S$ be a minimal ideal and suppose $\theta \in J$. Suppose $k \in \mathbf{N}$ and let $\mathcal{A}$ be any strongly partition regular family of configurations of cardinality $k$ in $\mathcal{G}_0$ that is closed under disjoint shifts. Let $N \in \mathbf{N}$ and let $V \subset (X^X)^k$ consist of all $k$-tuples $(T_{A_1}, T_{A_2}, \cdots, T_{A_k})$, where $\{A_1, \cdots, A_k\} \in \mathcal{A}$ with $A_i \in \mathcal{G}_{N+1}$, $1 \le i \le k$. Then $(\theta, \cdots, \theta) \in \overline{V}$.

**Proof.** We must show that for all $M, t \in \mathbf{N}$ and any choice of $\gamma_1, \cdots, \gamma_t \in X$, $E_1, \cdots, E_t \in \mathcal{G}_0$, there exists $\{A_1, \cdots, A_k\} \in \mathcal{A}$ such that $A_m \in \mathcal{G}_{M+1}$, $1 \le m \le k$, and

$$T_{A_m} \gamma_j(E_j) = \theta \gamma_j(E_j),\ 1 \le m \le k,\ 1 \le j \le t.$$

Let $l$ be large enough that $E_i \subset \{1, 2, \cdots, l\}^2$, $1 \le i \le t$. By Proposition 1.4 the set

$$\mathcal{B}_l = \big\{E : \gamma_j(E \cup A) = \theta \gamma_j(A),\ 1 \le j \le t,\ A \subset \{1, 2, \cdots, l\}^2\big\}$$

is a strongly piecewise syndetic set. This implies that the (perhaps larger) set

$$\mathcal{B} = \big\{E : \gamma_j(E \cup E_j) = \theta \gamma_j(E_j),\ 1 \le j \le t\big\}$$

is a strongly piecewise syndetic set as well, so that by Proposition 1.3, $\mathcal{B}$ contains a configuration $\{A_1, \cdots, A_k\} \in \mathcal{A}$ with $A_m \in \mathcal{G}_{N+1}$, $1 \le m \le k$. We are done.

□

Taking $\theta$ to be a minimal *idempotent* in $S$, we get a notion of largeness for subsets of $\mathcal{G}$ that will be useful for us.

**Definition 1.7** A family $\mathcal{C} \subset \mathcal{G}$ is said to be *strongly central* if there exists a minimal idempotent $\theta \in S$ such that $\theta 1_{\mathcal{C}}(\emptyset) = 1$.

As suggested by the name, strongly central sets in $(\mathcal{G}, \cup)$ are also *central sets* (see [F, Chapter 8] and [BH, Section 6]), but not all central sets in this semigroup are strongly central, nor is the notion of centrality especially useful in our current context.

**Proposition 1.8** Strongly central sets are strongly piecewise syndetic. Moreover, if $r \in \mathbf{N}$ and $\mathcal{G} = \bigcup_{i=1}^{r} C_i$ then some cell $C_j$ is strongly central.

**Proof.** See the proof the Corollary 1.5, only take $\theta$ to be a minimal *idempotent.*

□

We now move directly to a combinatorial application which is a 2-dimensional version of our main theorem, cast in the language of subsets of $\mathbf{N}^2$. Suppose $(A_i)_{i=1}^{\infty} \subset \mathcal{G}_1$ is a sequence of pairwise disjoint sets, and that $(B_i)_{i=1}^{\infty} \subset \mathcal{F}$ is a sequence of pairwise disjoint sets such that, furthermore, $(B_i \times B_j) \cap A_k = \emptyset$ for $i, j, k \in \mathbf{N}$. For aesthetic reasons, we shall also require that there exist an increasing sequence $(M_i)_{i=1}^{\infty} \subset \mathbf{N}$ such that

$$B_i \subset \{M_{i-1} + 1, \cdots, M_i\} \text{ and } A_i \subset \{1, \cdots, M_i\}^2 \setminus \{1, \cdots, M_{i-1}\}^2. \tag{1.2}$$

For $N \in \mathbf{N}$, let $\mathcal{M}_N$ be the set of $N \times N$ matrices with entries coming from $\{0,1\}$. Let $\mathcal{M} = \bigcup_{N=1}^{\infty} \mathcal{M}_N$. For $N \in \mathbf{N}$ and $M = (m_{ij}) \in \mathcal{M}_N$, let

$$K(M) = (A_1 \cup A_2 \cup \cdots \cup A_N) \cup \bigcup_{m_{ij}=1} (B_i \times B_j). \tag{1.3}$$

Letting $N$ go over all of $\mathbf{N}$, we get a function $K : \mathcal{M} \to \mathcal{G}$. We shall refer to the range of any function $K$ which arises in this manner, which may be represented as a sequence $(C_M)_{M \in \mathcal{M}}$, where $C_M = K(M)$, as an *$\mathcal{M}$-system.* We shall now prove an infinitary theorem concerning $\mathcal{M}$-systems.

**Theorem 1.9** Let $C \subset \mathcal{G}$ be strongly central. Then $C$ contains an $\mathcal{M}$-system.

**Proof.** Let $\theta \in S$ be a minimal idempotent with $\theta 1_C(\emptyset) = 1$. Consider the family of configurations

$$\mathcal{A}_1 = \big\{ \{A, A \cup B \times B\},\ A \in \mathcal{G},\ B \in \mathcal{F},\ A \cap (B \times B) = \emptyset \big\}$$

in $\mathcal{G}$. $\mathcal{A}_1$ is a strongly partition regular family, as we have noted previously. Furthermore, one easily sees that $\mathcal{A}_1$ is closed under disjoint shifts. Thus, if $V$ is the set of ordered pairs $\big\{ \big(T_A, T_{A \cup (B \times B)}\big) : \{A, A \cup (B \times B)\} \in \mathcal{A}_1 \big\}$ in $(X^X)^2$, by Theorem 1.6 we have $(\theta, \theta) \in \overline{V}$. In particular, we may select $A_1 \in \mathcal{G}_1$ and $B_1 \in \mathcal{F}$ such that $\{A_1, A_1 \cup (B_1 \times B_1)\} \in \mathcal{A}_1$ and such that $T_{A_1}$ and $T_{A_1 \cup (B_1 \times B_1)}$ are close enough to $\theta$ to ensure that

$$\begin{aligned}
&1_C(A_1) = T_{A_1} 1_C(\emptyset) = \theta 1_C(\emptyset) = 1, \\
&\theta 1_C(A_1) = T_{A_1} \theta 1_C(\emptyset) = \theta^2 1_C(\emptyset) = \theta 1_C(\emptyset), \\
&1_C(A_1 \cup (B_1 \times B_1)) = T_{A_1 \cup (B_1 \times B_1)} 1_C(\emptyset) = \theta 1_C(\emptyset) = 1, \text{ and} \\
&\theta 1_C(A_1 \cup (B_1 \times B_1)) = T_{A_1 \cup (B_1 \times B_1)} \theta 1_C(\emptyset) = \theta^2 1_C(\emptyset) = \theta 1_C(\emptyset) = 1.
\end{aligned}$$

Let $M_1$ be the smallest integer such that $A_1$ and $A_1 \cup (B_1 \times B_1)$ are each contained in $\{1, \cdots, M_1\}^2$.

Let now $\mathcal{A}_2$ be the family of configurations of the form

$$\{A \cup p_1(B), A \cup p_2(B), \cdots, A \cup p_8(B)\},$$

where $p_1(B), \cdots, p_8(B)$ are the 8 set-polynomials having empty constant term which consist of the union of some subset of the three set-polynomials $\{B \times B_1, B_1 \times B, B \times B\}$. $\mathcal{A}_2$ is a strongly partition regular family of configurations closed under disjoint shifts. Hence, by Theorem 1.4, if we let $V \subset (X^X)^8$ consist of all 8-tuples $(T_{D_1}, \cdots, T_{D_8})$, where $\{D_1, \cdots, D_8\} \in \mathcal{A}_2$ and each $D_i \in \mathcal{G}_{M+1}$, we have $(\theta, \cdots, \theta) \in \overline{V}$. Therefore, we may select $A_2 \in \mathcal{G}_{M+1}$ and $B_2 \in \mathcal{F}$ (containing no element less than $M_1 + 1$) such that

$$\begin{aligned}
&1_C(A_2 \cup p_i(B_2) \cup E) = T_{A_2 \cup p_i(B_2)} 1_C(E) = \theta 1_C(E) = 1 \text{ and} \\
&\theta 1_C(A_2 \cup p_i(B_2) \cup E) = T_{A_2 \cup p_i(B_2)} \theta 1_C(E) = \theta^2 1_C(E) = \theta 1_C(E) = 1, \\
&1 \leq i \leq 8,\ E \in \{A_1, A_1 \cup (B_1 \times B_1)\}.
\end{aligned}$$

Let $M_2$ be the smallest integer such that $A_2$ and $B_2 \times B_2$ lie in $\{1, \cdots, M_2\}^2$.

Let us take account of how the proof is progressing. We now have the sets $A_1$ and $A_1 \cup (B_1 \times B_1)$ in $\mathcal{C}$. These are exactly the images of the $1 \times 1$ matrices (0) and (1) respectively under the map $K$ of (1.3). We also have

$$\{A_2 \cup A_1 \cup E : E \text{ is a union of some of } \{(B_1 \times B_1), (B_1 \times B_2), (B_2 \times B_1), (B_2 \times B_2)\}\} \subset C.$$

This family consists precisely of the images of the members of $\mathcal{M}_2$ under the map $K$ as defined by (1.3). Moreover, we may continue in this fashion, utilizing the idempotence of $\theta$. Namely, having chosen $A_1, \cdots, A_t \in \mathcal{G}_1$ and $B_1, \cdots, B_t \in \mathcal{F}$ with $1_C(K(M)) = \theta 1_C(K(M)) = 1$ for all $M \in \mathcal{M}_t$, where $K$ is given by (1.3), and $M_1 < M_2 < \cdots < M_t$ (such that (1.2) holds), we may find $A_{t+1} \in \mathcal{G}_{M_t+1}$ and $B_{t+1} \in \mathcal{F}$ (none of whose members are less than $M_t + 1$), such that

$$\begin{aligned}
&1_C(A_{t+1} \cup p(B_{t+1}) \cup E) = T_{A_{t+1} \cup p(B_{t+1})} 1_C(E) = \theta 1_C(E) = 1 \text{ and} \\
&\theta 1_C(A_{t+1} \cup p(B_{t+1}) \cup E) = T_{A_{t+1} \cup p(B_{t+1})} \theta 1_C(E) = \theta^2 1_C(E) = \theta 1_C(E) = 1
\end{aligned}$$

for all $E \in K(\mathcal{M}_t)$ and all set polynomials $p(B)$ which are a union of some (possibly none) of the monomials $(B \times B)$, $(B_j \times B)$ and $(B \times B_j)$, $1 \leq j \leq t$. We let $M_{t+1}$ be the smallest integer such that $\{1, \cdots, M_{t+1}\}^2$ contains $K(\mathcal{M}_{t+1})$ (where, again, $K$ is defined by (1.2); we keep mentioning it because we are building the map $K$ as we go). Notice as well that now $1_C(K(M)) = \theta 1_C(K(M)) = 1$ for all $M \in \mathcal{M}_{t+1}$, so we can continue, thus completing the proof.

□

We now will change our focus slightly. Suppose we are given an increasing sequence $(R_i)_{i=1}^{\infty}$ of natural numbers, and a sequence of sets non-empty sets $B_i \subset \{R_{i-1}+1, R_{i-1}+$

$2, \cdots, R_i\}$. For every $(l,m) \in \mathbf{N} \times \mathbf{N}$, let $a_{lm}$ be the symbol $x_{ij}$ if $(l,m) \in B_i \times B_j$. Otherwise, let $a_{lm} \in \{0,1\}$. Then $V(x_{ij}) = (a_{lm})_{l,m \in \mathbf{N}}$ is an $\mathbf{N} \times \mathbf{N}$ matrix whose entries come from the set $\{0,1\} \cup \{x_{ij} : i,j \in \mathbf{N}\}$. Moreover, for fixed $m \in \mathbf{N}$, the matrix $V_m(x_{ij}) = (a_{lm})_{l,m=1}^{R_m}$ is an $R_m \times R_m$ matrix whose entries come from the set $\{0,1\} \cup \{x_{ij} : 1 \leq i,j \leq m\}$.

A matrix of this type induces a natural injection $(t_{ij})_{i,j=1}^{l} \to V_m(t_{ij})$ from $\mathcal{M}_l$ to $\mathcal{M}_{R_l}$. Namely, $V_m(t_{ij})$ is the $R_m \times R_m$ matrix which results by substituting $t_{ij}$ for the symbol $x_{ij}$ in the matrix $V_m(x_{ij}) = (a_{ij})_{i,j=1}^{R_m}$ constructed above. Hence, the $\mathbf{N} \times \mathbf{N}$ matrix $V(x_{ij}) = (a_{lm})_{l,m \in \mathbf{N}}$, together with the sequence $(R_m)_{m=1}^{\infty}$, induces such maps for all $m$; in other words, induces an injection of $\mathcal{M}$ into $\mathcal{M}$ (which takes $m \times m$ matrices to $R_m \times R_m$ matrices). We call the image of such a map an $\mathcal{M}$*-ring*. Specifically, the $\mathcal{M}$-ring generated by $(R_m)_{m=1}^{\infty}$ and the variable matrix $V(x_{ij}) = (a_{lm})$.

Hence for any $\mathcal{M}$-ring $\mathcal{N}$, there is an associated bijection $\varphi_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$, where $\varphi$ arises as outlined above. We note that if $\mathcal{R}$ is another $\mathcal{M}$-ring and $\varphi_{\mathcal{R}} : \mathcal{M} \to \mathcal{R}$ the associated bijection, then $\varphi_{\mathcal{R}} \circ \varphi_{\mathcal{N}}$ is again a map arising in the fashion outlined above, so that $\varphi_{\mathcal{R}}(\mathcal{N})$ is again an $\mathcal{M}$-ring, called a *subring* of $\mathcal{R}$.

**Theorem 1.10** Let $\mathcal{N}$ be an $\mathcal{M}$-ring. For any finite partition $\mathcal{N} = \bigcup_{i=1}^{r} C_i$, one of the cells $C_i$ contains a subring of $\mathcal{N}$.

**Proof.** First of all, assume that the result is known for $\mathcal{N} = \mathcal{M}$. Any finite coloring of $\mathcal{N}$ induces a coloring of $\mathcal{M}$ via the bijection $\varphi_{\mathcal{N}}$. Extracting a monochromatic $\mathcal{M}$-ring $\mathcal{R}$ for this induced coloring, $\varphi_{\mathcal{N}}(\mathcal{R})$ is a subring of $\mathcal{N}$ that is monochromatic for the original coloring. Hence we may assume without loss of generality that $\mathcal{N} = \mathcal{M}$.

Suppose, then, that $\mathcal{M} = \bigcup_{i=1}^{r} C_i$. We will induce an $r$-cell partition $\mathcal{G}_1 = \bigcup_{i=1}^{r} D_i$ as follows: for $E \in \mathcal{G}_1$, let $N$ be the smallest integer such that $E \subset \{1, \cdots, N\}^2$. Then let $E \in D_i$ if and only if the $N \times N$ matrix $(a_{ij})$, where $a_{ij} = 1$ if $(i,j) \in E$ and $a_{ij} = 0$ otherwise, is in $C_i$.

One of the cells $D_b$, where $1 \leq b \leq r$, must be strongly central and therefore by Theorem 1.9 contains an $\mathcal{M}$-system generated by a sequence $(A_i)_{i=1}^{\infty} \subset \mathcal{G}_1$ and a sequence $(B_i)_{i=1}^{\infty} \subset \mathcal{F}$. Furthermore there is an associated increasing sequence $(M_l)_{l=1}^{\infty} \subset \mathbf{N}$ such that $M_l$ is the least integer satisfying $A_l \cup (B_l \times B_l) \subset \{1, \cdots, M_l\}^2$. Put $B'_i = B_{2i-1}$, $A'_i = A_{2i} \cup A_{2i-1} \cup B_{2i} \times B_{2i}$, and $N_i = M_{2i}$, $i \in \mathbf{N}$. Let $V(x_{ij}) = (a_{ij})_{i,j \in \mathbf{N}}$ be the variable matrix obtained by letting $a_{kl} = x_{ij}$ if $(k,l) \in B'_i \times B'_j$, $a_{kl} = 1$ if $(k,l) \in \bigcup_{i=1}^{\infty} A'_i$, and $a_{kl} = 0$ otherwise.

We claim that the $\mathcal{M}$-ring $\mathcal{R}$ generated by $(N_i)_{i=1}^{\infty}$ and $V(x_{ij})$ is contained in $C_b$. To see this, let $l \in \mathbf{N}$ be arbitrary. We will show that $\varphi_{\mathcal{R}}(\mathcal{M}_l) \subset C_b$.

By hypothesis, every set having the form

$$E = \big(A'_1 \cup A'_2 \cup \cdots \cup A'_l\big) \cup \Big( \bigcup_{r_{ij}=1} (B'_i \times B'_j) \Big), \ (r_{ij}) \in \mathcal{M}_l \tag{1.4}$$

lies in $D_b$. Moreover, every set of this form has $N_l = M_{2l}$ as the least integer such that $\{1, \cdots, N_l\}^2$ contains it. (Recall that $A'_l = A_{2l-1} \cup A_{2l} \cup B_{2l} \times B_{2l}$.) That means that every $(a_{ij}) \in \mathcal{M}_{N_l}$ having the property that $a_{ij} = 1$ if and only if $(i,j)$ lies in a given set of the form (1.4) lies in $C_b$. In other words, $\varphi_{\mathcal{R}}(\mathcal{M}_l) \subset C_b$.

□

## 2. Infinitary polynomial Hales-Jewett theorem.

In this section we shall extend (without giving full details of the proof) Theorem 1.10 in two senses. First note that the $\mathcal{M}$-rings of the previous section could well be called $\mathcal{M}^{(2,2)}$-rings. The first 2 in this proposed superscript is owing to the fact that an $\mathcal{M}$-ring consists of 2-dimensional matrices, that is, indexed by $\{1,\cdots,N\}^2$ for some $N\in\mathbf{N}$. One might just as easily consider matrices $(a_{ijk})$ indexed by $\{1,\cdots,N\}^3$, or more generally, indexed by $\{1,\cdots,N\}^l$, $l\in\mathbf{N}$. The second 2 refers to the cardinality of the set from which the entries of the matrices are drawn. That set is $\{0,1\}$. One might consider taking a set of cardinality $k$, such as $\{0,1,\cdots,k-1\}$, as the set from which those entries are drawn.

As a matter of fact, neither of these considerations poses any obstacle to obtaining correspondingly more general versions of Theorems 1.9 and 1.10. We will, in this section, give a few of the details on the formulation of a more general version of Theorem 1.10, and its proof. For $R,l,k\in\mathbf{N}$, we will denote by $\mathcal{M}_R^{(l,k)}$ the set of all functions (matrices) $A:\{1,\cdots,R\}^l\to\{0,1,\cdots,k-1\}$. Then we let $\mathcal{M}^{(l,k)}=\bigcup_{R=1}^\infty\mathcal{M}_R^{(l,k)}$. We now proceed to define $\mathcal{M}^{(l,k)}$-rings. Suppose we are given an increasing sequence $(R_i)_{i=1}^\infty$ (let $R_0=0$) of natural numbers, and a sequence of non-empty sets $(B_i)_{i=1}^\infty$ with $B_i\subset\{R_{i-1}+1,R_{i-1}+2,\cdots,R_i\}$, $i\in\mathbf{N}$. For every $(i_1,\cdots,i_l),(j_1,\cdots,j_l)\in\mathbf{N}^l$, let $a_{i_1i_2\cdots i_l}$ be the symbol $x_{j_1j_2\cdots j_l}$ if $(i_1,i_2,\cdots,i_l)\in B_{j_1}\times B_{j_2}\times\cdots\times B_{j_l}$. Otherwise, let $a_{i_1i_2\cdots i_l}\in\{0,1,\cdots,k-1\}$. Then $V(x_{j_1j_2\cdots j_l})=(a_{i_1i_2\cdots i_l})_{i_1,i_2,\cdots,i_l\in\mathbf{N}}$ is a matrix indexed by $\mathbf{N}^l$ whose entries come from the set $\{0,1,\cdots,k-1\}\cup\{x_{j_1j_2\cdots j_l}:j_1,\cdots,j_l\in\mathbf{N}\}$. Moreover, for fixed $m\in\mathbf{N}$, the matrix $V_m(x_{j_1j_2\cdots j_l})=(a_{i_1i_2\cdots i_l})_{i_1,\cdots,i_l=1}^{R_m}$ is a matrix indexed by $\{1,\cdots,R_m\}^l$ whose entries come from the set $\{0,1,\cdots,k-1\}\cup\{x_{j_1j_2\cdots j_l}:1\le j_1,\cdots,j_l\le m\}$. A matrix of this type induces an injection of $\mathcal{M}_m^{(l,k)}$ into $\mathcal{M}_{R_m}^{(l,k)}$. Letting $m$ range over $\mathbf{N}$, the matrix $V(x_{j_1j_2\cdots j_l})$ induces a map from $\mathcal{M}^{(l,k)}$ to $\mathcal{M}^{(l,k)}$. We call the image of such a map an $\mathcal{M}^{(l,k)}$*-ring*. Specifically, the $\mathcal{M}^{(l,k)}$-ring generated by $(R_i)_{i=1}^\infty$ and the variable matrix $V(x_{j_1j_2\cdots j_l})=(a_{i_1i_2\cdots i_l})_{i_1,i_2,\cdots,i_l\in\mathbf{N}}$. Subrings of $\mathcal{M}^{(l,k)}$-rings may be defined much as they were for $\mathcal{M}$-rings in Section 1.

We want to extend Theorem 1.10 of the previous section to $\mathcal{M}^{(l,k)}$-rings. The natural way to accomplish this is to first extend Theorem 1.9 to more general types of systems, similar to $\mathcal{M}$-systems, that are subsets of $\mathcal{G}^{(l,k)}=\left(\mathcal{F}(\mathbf{N}^l)\right)^k$, the semigroup of all $k$-tuples of finite subsets of $\mathbf{N}^l$. All of the definitions (e.g. strong piecewise syndeticity, strong partition regularity, strong centrality) and propositions in Section 1 extend quite easily to $\mathcal{G}^{(l,k)}$. The trickiest part of the adaptation owes itself to the fact that Theorem 0.2 is not formulated in a manner that lends itself easily to use in this new environment. Hence we give now a version that does.

**Theorem 2.1** Let $l,k,t\in\mathbf{N}$ and let $p_{i,j}(X)$, $1\le i\le t$, $1\le j\le k$ be set-polynomials over $\mathbf{N}^l$ whose constant terms are empty. Let $H\subset\mathbf{N}$ be any finite set and let $r\in\mathbf{N}$. There exists a finite set $M\subset\mathbf{N}$, with $M\cap H=\emptyset$, having the property that if $\mathcal{F}(\mathbf{N}^l)^k=\bigcup_{i=1}^r C_i$ then there exists some $d$ with $1\le d\le r$, some non-empty $N\subset M$, and some sets $A_1,A_2,\cdots,A_k\subset\bigcup_{i=1}^t\bigcup_{j=1}^k p_{i,j}(M)$, such that $A_s\cap p_{i,j}(N)=\emptyset$, $1\le i\le t$, $1\le j,s\le k$, and

$$\big\{\big(A_1 \cup p_{i,1}(N), A_2 \cup p_{i,2}(N), \cdots, A_k \cup p_{i,k}(N)\big) : 1 \le i \le t\big\} \subset C_d.$$

Although the above formulation of the polynomial Hales-Jewett theorem is not given explicitly in [BL2], it is implicit in the exposition, and hence we shall omit the proof (which, at any rate, follows quite easily from Theorem 0.2; the key to seeing this is to identify the $k$-tuple of sets $(A_1, \cdots, A_{k-1})$ in $\mathbf{N}^l$ with the set $\big((\{1\}\times A_1)\cup(\{2\}\times A_2)\cup\cdots\cup(\{k-1\}\times A_{k-1})\big)$ in $\mathbf{N}^{l+1}$ and consider the family of set-polynomials $\big\{\{i\} \times p_j(X) : \ 1 \le i \le k-1, \ 1 \le j \le t)\big\}$.

Supposing one has the more general form of Theorem 1.9, one must still do something to get from there to a more general form of Theorem 1.10. In Section 1 this was accomplished quite easily, as there is a natural correspondence between subsets of $\{1, \cdots, N\}^2$ and $N \times N$ matrices whose entries are drawn from $\{0, 1\}$. The situation here is only slightly more complicated; there is a natural correspondence between $k$-tuples of subsets of $\{1, \cdots, N\}^l$ and $N \times N \times \cdots \times N$ ($l$ times) "matrices" whose entries are drawn from $\{0, 1, \cdots, 2^k - 1\}$.

In order to better elucidate the argument (in getting to the more general form of Theorem 1.10), we shall examine in some detail a *finitary* case of moving from $k$-tuples of sets to matrices. For convenience we again consider a case where $l = 2$. Let us denote, for $M > N$,

$$\begin{aligned} &L(\{N+1, \cdots, M\}) \\ =&\big(\{N+1, N+2, \cdots, M\} \times \{1, 2, \cdots, M\}\big) \cup \big(\{1, 2, \cdots, N\} \times \{N+1, N+2, \cdots, M\}\big). \end{aligned}$$

Notice that $L(\{N+1, \cdots, M\})$ is shaped like an $L$. The following corollary to Theorem 2.1 concerns itself with matrices indexed not by squares in the plane but by such $L$-shaped sets.

**Corollary 2.2** Let $l \in \mathbf{N}$ and let $p_1(X), \cdots, p_t(X)$ be set-polynomials over $\mathbf{N}^l$ whose constant terms are empty. Let $N, r \in \mathbf{N}$. Suppose that $p_i(A) \cap p_j(B) = \emptyset$ for $i \ne j$ and every pair of sets $A, B$ such that $(A \cup B) \cap \{1, \cdots, N\} = \emptyset$. Then for every $k, r \in \mathbf{N}$ there exists $M > N$ such that for any function

$$c : \{0, 1, \cdots, k-1\}^{L(\{N+1, \cdots, M\})} \to \{1, \cdots, r\},$$

there exists some $v \in \{0, 1, \cdots, k-1\}^{L(\{N+1, \cdots, M\})}$ and some set $B \subset \{N+1, \cdots, M\}$ such that for every $u_1, u_2 \in \{0, 1, \cdots, k-1\}^{L(\{N+1, \cdots, M\})}$ that agree with $v$ off of $\bigcup_{j=1}^{t} p_j(B)$, and with $u_1$ and $u_2$ each constant on every $p_j(B)$, $c(u_1) = c(u_2)$.

The content of Corollary 2.2 is: if eventually the set polynomials $p_i$ have pairwise disjoint ranges then for any $r$-coloring of large enough ($L$-shaped) matrices whose coordinates are letters from the alphabet $\{0, 1, \cdots, k-1\}$, it is possible to choose a set $B$ and a large enough matrix such that the color of the matrix remains constant over all possible values of the letters occuring on each $p_j(B)$ (provided that this letter is constant over each $p_j(B)$).

An example of this: say $A \subset \{1, \cdots, N\}$ and one has the three set polynomials $p_1(B) = A \times B$, $p_2(B) = B \times A$, and $p_3(B) = B \times B$. Then for $B \cap \{1, \cdots, N\} = \emptyset$, the sets $p_i(B)$ are pairwise disjoint. Hence, for any finite coloring of the matrices over the set $L(\{N+1, \cdots, M\})$, where $M$ is large enough, using the alphabet $\{0, 1, \cdots, k-1\}$, there exists some matrix $v$ and a set $B$ such that for any replacement of the letters in $v$ by a letter $i_1$ on $A \times B$, $i_2$ on $B \times B$, and $i_3$ on $B \times A$, the color of the resulting matrix does not depend on $i_1, i_2$, or $i_3$.

As for why Corollary 2.2 follows from Theorem 2.1, consider first of all that, given $k$, if we show Corollary 2.2 holds for $k$ replaced by something bigger than $k$ (say, $2^k$), then it trivially holds for $k$ as well (we can just consider colorings that identify certain letters). As mentioned earlier, $k$-tuples of finite subsets of $\mathbf{N}^2$ may be identified with $\mathbf{N} \times \mathbf{N}$ matrices with entries from the set $\{0, 1, \cdots, 2^k - 1\}$, all but finitely many of whose entries are zero. (Given such a $k$-tuple $(A_1, \cdots, A_k)$ and $x \in \mathbf{N}^2$, one can let $a_x$ be the number whose binary representation is $1_{A_1}(x) 1_{A_2}(x) \cdots 1_{A_k}(x)$.) Using this identification and considering the set of all polynomial $k$-tuples $\big(q_{i,1}(X), \cdots, q_{i,k}(X)\big)$, where each $q_{i,j}$ is a union of some of the $p_i(X)$'s, Theorem 2.1 may be used to get Corollary 2.2 with $k$ replaced by $2^k$.

Here now is our main theorem.

**Theorem 2.3** Let $l, k \in \mathbf{N}$ and let $\mathcal{N}$ be an $\mathcal{M}^{(l,k)}$-ring. For any finite partition $\mathcal{N} = \bigcup_{i=1}^r C_i$, one of the cells $C_i$ contains a subring of $\mathcal{N}$.

In conclusion, let us prove Theorem 0.2 from Theorem 2.3. Let $\mathcal{W}^{(k)} = \bigcup_{i=1}^r C_i$. $\mathcal{W}^{(k)}$ is an $\mathcal{M}^{(1,k)}$-ring, so by Theorem 2.3 there exists a monochromatic $\mathcal{M}^{(1,k)}$-ring $\mathcal{N} \subset \mathcal{W}^{(k)}$. All we must do is to show that $\mathcal{N}$ has the correct form, so we simply look at its structure.

There exists an increasing sequence $(R_i)_{i=1}^\infty \subset \mathbf{N}$ (with $R_0 = 0$) and a sequence of non-empty sets $(B_i)_{i=1}^\infty$, with $B_i \subset \{R_{i-1}+1, \cdots, R_i\}$, and a sequence of symbols $(a_i)_{i=1}^\infty$, with $a_i = x_j$ if $i \in B_j$ for some $j$ and $a_i \in \{0, 1, \cdots, k-1\}$ otherwise. $\mathcal{N}$ consists of all 1-dimensional matrices (i.e. words) $b_1 b_2 \cdots b_{R_M}$, as $M$ ranges over $\mathbf{N}$, $s_1, \cdots, s_M$ range over $\{0, 1, \cdots, k-1\}$, and $b_i = s_j$ if $a_i = x_k$ for some $j$ and $b_i = a_i$ otherwise.

For $m \in \mathbf{N}$, let $w_m(x)$ be the variable word formed by taking the word $a_{R_{m-1}+1} \cdots a_{R_m}$ and replacing all occurences of $x_m$ by $x$. One now easily checks that

$$\mathcal{N} = \big\{ w_1(i_1) w_2(i_2) \cdots w_M(i_M) : M \in \mathbf{N},\ i_j \in \{0, 1, \cdots, k-1\},\ 1 \le j \le M \big\}.$$

In other words, the monochromatic configuration found by Theorem 2.3 is precisely of the type needed for Theorem 0.2.

REFERENCES

[BBH] V. Bergelson, A. Blass and N. Hindman, Partition theorems for spaces of variable words, *Proc. Lond. Math. Soc. (3)* **68** (1994), 449-476.

[BH] V. Bergelson and N. Hindman, Nonmetrizable topological dynamics and Ramsey theory, *Trans. AMS* **320** (1990), 293-320.

[BL1] V. Bergelson and A. Leibman, Polynomial extensions of van der Waerden's and Szemerédi's theorems, *Journal of AMS* **9** (1996), 725-753.

[BL2] V. Bergelson and A. Leibman, Set polynomials and a polynomial extension of the Hales-Jewett theorem, *Ann. Math.* **150** (1999).

[CS] T. Carlson and S. Simpson, A dual form of Ramsey's theorem, *Adv. in Math.* **53** (1984), 265-290.

[E] R. Ellis, *Lectures on topological dynamics*, Benjamin, New York, 1969.

[F] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, 1981.

[FK] H. Furstenberg and Y. Katznelson, Idempotents in compact semigroups and Ramsey theory, *Israel J. Math.* **68** (1990), 257-270.

[HJ] A.W. Hales and R.I. Jewett, Regularity and positional games, *Trans. AMS* **106** (1963), 222-229.

[H] N. Hindman, Finite sums from sequences within cells of a partition of **N**, *J. Combinatorial Theory* (Series A) **17** (1974) 1-11.

[HM] N. Hindman and R. McCutcheon, VIP-systems in partial semigroups. *Disc. Math.*, to appear.

[HS] N. Hindman and D. Strauss, *Algebra in the Stone-Čech Compactification: Theory and Applications*, de Gruyter, Berlin, 1998.

[M] R. McCutcheon, An infinitary polynomial van der Waerden theorem, *J. Comb. Th. Series A* **86** (1999), 214-231.

[vdW] B. L. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieuw Arch. Wisk.* **15** (1927), 212-216.

Randall McCutcheon
Department of Mathematics
University of Maryland
College Park, MD 20742
randall@math.umd.edu