An Infinitary Polynomial van der Waerden Theorem

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0. Introduction

Our subject is *infinitary Ramsey theory*, specifically the existence of monochromatic structures of infinite cardinality for finite colorings of semigroups. Specifically, we shall prove an infinitary version of a recent polynomial extension of van der Waerden's theorem due to Bergelson and Leibman ([BL1]). Alternatively, this theorem may be viewed as a polynomial version of an infinitary van der Waerden type result of Furstenberg (Proposition 8.21 in [F]). Then we show that our methods allow for an extension of this result to so-called *VIP-systems* in general countable abelian semigroups.

The subject matter encompassed by "infinitary Ramsey theory" may be well illustrated by an example. Schur's theorem ([S]) states that if $\mathbf{N} = \bigcup_{i=1}^r C_i$ then some cell C_j contains two distinct natural numbers together with their sum, namely a configuration of the form $\{m, n, m+n\}$. A finitary extension of Schur's theorem (see [GRS], where it is attributed to Folkman), guarantees in one cell a configuration consisting of k distinct natural numbers together with all their sums (without repeats). The infinitary version of Schur's theorem, due to Hindman ([H]), is a deeper result than the finitary versions. It states that in one cell of any finite partition of $\mathbf{N}$ one can always find an *infinite* sequence $(n_i)_{i=1}^\infty \subset \mathbf{N}$ together with all of its finite sums (without repeats), namely

$$FS\big((n_i)_{i=1}^\infty\big) = \{n_{i_1} + n_{i_2} + \cdots + n_{i_k} : i_1 < i_2 < \cdots < i_k,\ k \in \mathbf{N}\}. \tag{0.1}$$

(FS stands for "finite sums".)

Any set having the form (0.1) is called an *IP-set*. It is convenient to represent an IP-set by an *$\mathcal{F}$-sequence* $(m_\alpha)_{\alpha\in\mathcal{F}}$, where $\mathcal{F}$ is the set of non-empty, finite subsets of $\mathbf{N}$. The $\mathcal{F}$-sequence generated by a sequence $(n_i)_{i=1}^\infty$ is given by $m_\alpha = \sum_{i\in\alpha} n_i$, $\alpha \in \mathcal{F}$, and one easily checks that $m_{\alpha\cup\beta} = m_\alpha + m_\beta$ whenever $\alpha\cap\beta = \emptyset$. Conversely, if an $\mathcal{F}$-sequence $(m_\alpha)_{\alpha\in\mathcal{F}} \subset \mathbf{N}$ satisfies $m_{\alpha\cup\beta} = m_\alpha + m_\beta$ for $\alpha\cap\beta=\emptyset$ then, letting $n_i = m_{\{i\}}$, the sets $FS\big((n_i)_{i=1}^\infty\big)$ and $\{m_\alpha : \alpha\in\mathcal{F}\}$ are equal. Hence, we will usually refer to any $\mathcal{F}$-sequence $(m_\alpha)_{\alpha\in\mathcal{F}}$ of natural numbers satisfying $m_{\alpha\cup\beta} = m_\alpha + m_\beta$ whenever $\alpha\cap\beta=\emptyset$ as an IP-set (even though, properly speaking, it is $\{m_\alpha : \alpha\in\mathcal{F}\}$ which is the IP-set).

Subsequences of $\mathcal{F}$-sequences $(x_\alpha)_{\alpha\in\mathcal{F}}$ arise by restricting α to a special type of subset of $\mathcal{F}$ having the same structure as $\mathcal{F}$. Namely, suppose $(\alpha_i)_{i=1}^\infty \subset \mathcal{F}$ has the property that $\alpha_i < \alpha_j$ (in the sense that every member of α_i is less than every member of α_j) for all $i < j$. Let $\mathcal{F}^{(1)}$ be the set of all finite unions of the α_i's. Then $\mathcal{F}^{(1)}$ is called an *IP-ring*. The correspondence $\beta \leftrightarrow \sum_{i\in\beta}\alpha_i$ is a union-preserving bijection between $\mathcal{F}$ and $\mathcal{F}^{(1)}$, and $(x_\alpha)_{\alpha\in\mathcal{F}^{(1)}}$ is said to be an IP-subsequence of $(x_\alpha)_{\alpha\in\mathcal{F}}$.

Van der Waerden's theorem ([vdW], see [F] or [GRS]) states that if the set of natural numbers $\mathbf{N} = \{1, 2, \cdots\}$ is partitioned into finitely many cells, $\mathbf{N} = \bigcup_{i=1}^r C_i$, one of the cells contains an arithmetic progression of length k for all k. In [BL1], Bergelson and Leibman extended this result about arithmetic progressions to include "polynomial progressions".

A special case of their theorem states that if $\{p_1(x), \cdots, p_k(x)\} \subset \mathbf{Z}[x]$ with $p_i(0) = 0$ for all i, then for any finite partition $\mathbf{N} = \bigcup_{i=1}^r C_i$, some cell C_j contains a configuration of the form $\{a, a + p_1(n), \cdots, a + p_k(n)\}$, where $n \neq 0$. One sees that van der Waerden's theorem corresponds to the case of linear polynomials in this "polynomial van der Waerden theorem". The following infinitary version of this theorem is a special case of Theorem 1.6 below.

Theorem A. Let $k \in \mathbf{N}$ and suppose $\{p_1(x), \cdots, p_k(x)\} \subset \mathbf{Z}[x]$ are polynomials having zero constant term. Let $(n_\alpha)_{\alpha \in \mathcal{F}}$ be an IP-set. If $r \in \mathbf{N}$ and $\mathbf{N} = \bigcup_{i=1}^r C_i$ then there exists an IP-ring $\mathcal{F}^{(1)}$, an IP-set $(a_\alpha)_{\alpha \in \mathcal{F}^{(1)}}$, and some j with $1 \leq j \leq r$ such that for all $\alpha \in \mathcal{F}^{(1)}$,

$$\{a_\alpha, a_\alpha + p_1(n_\alpha), \cdots, a_\alpha + p_k(n_\alpha)\} \subset C_j.$$

The linear case of Theorem A is due to Furstenberg (a special case of Proposition 8.21 in [F]). A version of this theorem for general countable semigroups was later obtained by Furstenberg and Katznelson (see the remark after Theorem 2.5 in [FK]). It states:

Theorem B. Let S be a countable semigroup, let $k \in \mathbf{N}$ and suppose that $G \subset S^k$ is a semigroup containing the diagonal $\Delta(S) = \{(x, x, \cdots, x) : x \in S\}$. If $I \subset G$ satisfies $(IG \cup GI) \subset I$ (that is, if I is a two-sided *ideal* in G), then for any finite coloring of S there exists a sequence $(\tilde{\gamma}_i)_{i=1}^\infty \subset I$ such that, writing $\tilde{\gamma}_i = (\gamma_{i,1}, \cdots, \gamma_{i,k})$, the set of products

$$\{\gamma_{i_1,j_1}\gamma_{i_2,j_2}\cdots\gamma_{i_t,j_t} : t \in \mathbf{N},\ i_1 < i_2 < \cdots < i_t\}$$

is monochromatic.

To retrieve the linear case of Theorem A from Theorem B, let $S = \mathbf{N}$, $I = \{(a, a + n_\alpha, a + 2n_\alpha, \cdots, a + (k-1)n_\alpha :\ a \in \mathbf{N},\ \alpha \in \mathcal{F}\}$ and $G = I \cup \Delta(S)$. Other special cases of Theorem B include the Hales-Jewett theorem ([HJ]) and an infinitary version of the Hales-Jewett theorem due to Carlson and Simpson ([CS]). In Section 2, we will prove a theorem (Theorem 2.3) which contains Theorem B as a special case. As an application of the results of Section 2, we will obtain a much more general version of Theorem A (Theorem 2.8).

The non-linear case of Theorem A does not follow from Theorem B, however, due to the fact that the set of k-tuples $(a + p_1(n), \cdots, a + p_k(n))$ whose coordinates form a polynomial progression of a given form, while being shift invariant, is not a semigroup. For example, $(a, a + x^k) + (b, b + y^k)$ cannot, as luck would have it, be of the form $(c, c + z^k)$ if $k \geq 3$.

The apparatus used in this paper is similar to that of [FK], but there is an important difference in methodology owing to the fact that in [FK] no (finitary) Ramsey-type theorems are employed. Therefore, the proofs of the infinitary theorems in [FK] provide new proofs of their finitary versions, whereas our method requires the finitary version in order to get the infinitary version. The finitary case of Theorem A, which we now state, is due to Bergelson and Leibman (it is an unstated combinatorial corollary to Corollary 1.9 in [BL1]).

Theorem C. Let $k \in \mathbf{N}$ and suppose $\{p_1(x), \cdots, p_k(x)\} \subset \mathbf{Z}[x]$ are polynomials having zero constant term. Let $(n_\alpha)_{\alpha\in\mathcal{F}}$ be an IP-set. For any finite coloring of $\mathbf{N}$ there exists $a \in \mathbf{N}$ and $\alpha \in \mathcal{F}$ such that

$$\big\{a, a + p_1(n_\alpha), \cdots, a + p_k(n_\alpha)\big\}$$

is monochromatic.

Our proof of Theorem A uses Theorem C and hence does not provide a new proof of Theorem C. By the same token, the proof of Theorem 2.3 (which extends Theorem B) below does not provide a new proof of Theorem B. In fact, in order to see that Theorem B follows from Theorem 2.3, one has to take for granted Theorem B's finitary version.

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1. The infinitary polynomial theorem.

Suppose that G is a semigroup and $E \subset G$. E is said to be *left syndetic* if there exist $s_1, \cdots, s_k \in G$ such that $G = s_1^{-1}E \cup \cdots \cup s_k^{-1}E$, where $s^{-1}E = \{t \in G : st \in E\}$. A notion of right syndeticity may be similarly defined. (Obviously in commutative semigroups, the two notions coincide, and we say merely that E is *syndetic*; similar remarks apply to the notions to come. We also remark that what we are calling left syndeticity is called right syndeticity by some authors. We choose "left" because it seems to us more natural; E is left syndetic in G if finitely many of its left shifts cover G. Our choice is also consistent with prevailing usage of "left" and "right" in regard to amenability.) $E \subset G$ will be called *left thick* if for every finite set $F \subset G$, there exists $g \in G$ such that $Fg = \{fg : f \in F\} \subset E$. One may check that a set is left thick if and only if its complement fails to be left syndetic. Finally, E is said to be *piecewise left syndetic* if there exist $s_1, \cdots, s_k \in G$ such that $(s_1^{-1}E \cup \cdots \cup s_k^{-1}E)$ is left thick. One may show that E is left piecewise syndetic if and only if E is the intersection of a left thick set with a left syndetic set.

Proposition 1.1 Let G be a semigroup and let $E \subset G$. If $A \subset G$ is left syndetic and for every finite set $F \subset A$ there exists $g \in G$ such that $Fg \subset E$ then E is left piecewise syndetic.

Proof. We know that there exist $s_1, \cdots, s_k \in G$ such that $(s_1^{-1}A \cup \cdots \cup s_k^{-1}A) = G$. We claim that $(s_1^{-1}E \cup \cdots \cup s_k^{-1}E)$ is left thick. Let $F = \{f_1, \cdots, f_t\}$ be a finite set. For every i, $1 \le i \le t$, there exists j_i, $1 \le j_i \le k$, such that $s_{j_i}f_i \in A$. Hence by hypothesis there exists $g \in G$ such that $\{s_{j_1}f_1g, \cdots, s_{j_t}f_tg\} \subset E$. That is, $Fg \subset (s_1^{-1}E \cup \cdots \cup s_k^{-1}E)$. □

Piecewise syndetic sets are important to us because of the following theorem. Let us say that a family $\mathcal{A}$ of finite subsets of a semigroup G is *left shift invariant* if $sA \in \mathcal{A}$ whenever $A \in \mathcal{A}$ and $s \in G$. Right shift invariance is defined similarly. $\mathcal{A}$ will be called *partition regular* if for every finite partition $G = \bigcup_{i=1}^r C_i$, some cell C_j contains a member of $\mathcal{A}$.

Theorem 1.2 Suppose that G is a semigroup and $\mathcal{A}$ is a partition regular, left and right shift invariant family of finite subsets of G. Then:

(a) For every finite partition $E = \bigcup_{i=1}^{r} C_i$ of a left thick set $E \subset G$, come cell C_j contains a member of $\mathcal{A}$.

(b) every left piecewise syndetic subset of G contains a member of $\mathcal{A}$.

Proof. (a) Suppose for every finite set $F \subset G$, there exists an r-coloring $\gamma_F : F \to \{1, \cdots, r\}$ for which there is no monochromatic member of $\mathcal{A}$. Extending the domain of each γ_F to G by setting $\gamma_F(g) = 0$ for $g \notin F$, $\{\gamma_F\}$ is a net in the compact space $\{0, 1, \cdots, r\}^G$ indexed by the family of finite subsets of G (directed by inclusion). Let γ be the limit of any convergent subnet. One easily checks that the range of γ is contained in $\{1, 2, \cdots, r\}$, and that moreover there is no γ-monochrome member of $\mathcal{A}$, a contradiction.

Hence there exists a finite set $F \subset G$ such that for any r-coloring of F, there exists a monochromatic member of $\mathcal{A}$. Suppose now $E = \bigcup_{i=1}^{r} C_i$ and choose $g \in G$ such that $Fg \subset E$. For $f \in F$, let $f \in D_i$ if and only if $fg \in C_i$. Then $F = \bigcup_{i=1}^{r} D_i$ and some D_j contains some $A \in \mathcal{A}$. Then $Ag \subset C_j$.

(b) Let $E \subset G$ be left piecewise syndetic. There exist $s_1, \cdots, s_k \in G$ such that $s_1^{-1}E \cup \cdots \cup s_k^{-1}E$ is left thick. By (a), therefore, some $s_i^{-1}E$ contains a member A of $\mathcal{A}$. In other words, $s_i A \subset E$.

□

A semigroup S is called a *compact right topological semigroup* if it is endowed with a topology with respect to which it is a compact Hausdorff space and with respect to which the map $t \to ts$ is continuous for all $s \in S$. (Notice the asymmetry of this condition: we do not assume that the map $t \to st$ is continuous for all s, and in general this will not be the case.) Recall that an element $t \in S$ is called an *idempotent* if $t^2 = t$.

Proposition 1.3. (see [E]) Any compact right topological semigroup S possesses an idempotent.

Proof. Let $\mathcal{M}$ denote the family of non-empty closed subsets $P \subset S$ for which $P^2 \subset P$. By Zorn's Lemma that $\mathcal{M}$ contains a minimal element P with respect to inclusion. Let $p \in P$. Then $Pp \subset P$ is compact (being the continuous image of a compact set), non-empty, and moreover $(Pp)^2 \subset P$, hence $Pp = P$. In particular the set $Q = \{q \in P : qp = p\} \subset P$ is non-empty and, being the intersection of the continuous inverse image of a singleton with P, closed. Furthermore $Q^2 \subset Q$, so that $Q = P$. That is, $qp = p$ for all $q \in P$. In particular, $p^2 = p$.

□

Let S be a compact right topological semigroup and let $J \subset S$ be non-empty and closed. If $SJ \subset J$ then J is said to be a *left ideal*. Any left ideal, itself being a compact right topological semigroup, contains an idempotent by Proposition 1.3. If J is a left ideal of S which is minimal among left ideals with respect to inclusion, then we call J a *minimal left ideal*, and any idempotent $\theta \in J$ is called a *minimal idempotent*. By Zorn's Lemma every compact right topological semigroup contains a minimal left ideal and hence a minimal idempotent.

We shall now apply the foregoing notions. Put $\mathbf{N}_0 = \mathbf{N} \cup \{0\} = \{0, 1, 2, \cdots\}$. Let $r \in \mathbf{N}$ be fixed and put $X = \{0,1\}^{\mathbf{N}_0}$. With the product topology, X is a compact. Let $\Omega = X^X$. With the product topology, Ω is compact (though not metrizable). This topology has a subbasis consisting of sets of the form $\{\theta \in \Omega : \theta\gamma(n) = i\}$, where $\gamma \in X$ and $i \in \{0,1\}$. The map $g \to g \circ f$ is continuous for all $f \in \Omega$. Hence Ω forms a compact right topological semigroup under composition. We can embed $\mathbf{N}$ in Ω as follows: for every $n \in \mathbf{N}$ and every $\gamma \in X$ (that is, for every function $\gamma : \mathbf{N}_0 \to \{0,1\}$), let $T^n\gamma \in X$ be given by the rule $T^n\gamma(m) = \gamma(m+n)$. This determines a map $T^n : X \to X$, and one may check that in fact $\{T^n : n \in \mathbf{N}\}$ is a semigroup consisting of continuous self-maps of X. We shall restrict our attention to the closure in Ω of this embedded copy of $\mathbf{N}$ (i.e. the enveloping semigroup of T),

$$S = \overline{\{T^n : n \in \mathbf{N}\}}.$$

We shall repeatedly make use of the fact that for any subsets $A, B \subset \mathbf{N}$, $(\overline{A})(\overline{B}) \subset \overline{A+B}$ (where we identify $A \subset \mathbf{N}$ with $\{T^n : n \in A\} \subset S$). A proof of this (which is valid for general semigroups) may be found below equation (2.3) in [FK]. In particular, taking $A = B = \mathbf{N}$, we get that S is a semigroup. (Actually, S is the Stone-Čech compactification of $\mathbf{N}$ (see, for example, [HS, Theorem 19.15]). An isomorphism between S and $\beta\mathbf{N}$, taken as the set of ultrafilters on $\mathbf{N}$, is given by the map which sends $\phi \in S$ to the ultrafilter $p_\phi = \big\{E \subset \mathbf{N} : \phi \in \overline{\{T^n : n \in E\}}\big\}$. This isomorphism preserves the semigroup operation as well (composition in S corresponds to addition in $\beta\mathbf{N}$).)

The following proposition consists of standard facts about the Stone-Čech compactification of a semigroup. For a proof of (an ultrafilter exposition of) part (b) (which is all we need in the sequel), for example, see [HS, Theorem 4.40]. We include a proof for completeness.

Proposition 1.4 Let $J \subset S$ be a minimal left ideal and let $\theta \in J$. If $t, l \in \mathbf{N}$ and $\gamma_1, \cdots, \gamma_t \in X$ then:

(a) the set $B_l = \big\{n : \theta\gamma_j(n+i) = \theta\gamma_j(i),\ 1 \le j \le t,\ 0 \le i \le l\big\}$ is syndetic.

(b) the set $P_l = \big\{n : \gamma_j(n+i) = \theta\gamma_j(i),\ 1 \le j \le t,\ 0 \le i \le l\big\}$ is piecewise syndetic.

Proof. (a) Suppose not. Then for every $n \in \mathbf{N}$ there exists $l_n \in \mathbf{N}$ such that for every $m \in \{l_n, l_n+1, \cdots, l_n+n\}$, $\theta\gamma_j(m+i) \ne \theta\gamma_j(i)$ for some j, $1 \le j \le t$, and some i, $0 \le i \le l$. Equivalently, for every b, $0 \le b \le n$, $T^{l_n}\theta\gamma_j(b+i) \ne \theta\gamma_j(i)$ for some j and some i, $0 \le i \le l$. Let ϕ be an accumulation point in S of $\{T^{l_n} : n \in \mathbf{N}\}$. Then for every $b \in \mathbf{N}_0$, $\phi\theta\gamma_j(b+i) \ne \theta\gamma_j(i)$ for some j and some i, $0 \le i \le l$. Since J is a minimal left ideal, and $S\phi\theta \subset J$ is a left ideal, we have $S\phi\theta = J$. In particular, there exists $\psi \in S$ such that $\theta = \psi\phi\theta$. Recall that $S = \overline{\{T^n : n \in \mathbf{N}\}}$. Hence we may choose $b \in \mathbf{N}$ such that T^b approximates ψ to the extent that $T^b\phi\theta\gamma_j(i) = \psi\phi\theta\gamma_j(i) = \theta\gamma_j(i)$, $1 \le j \le t$, $0 \le i \le l$. (By this we mean that T^b lies in the open set $\{\xi : \xi\phi\theta\gamma_j(i) = \psi\phi\theta\gamma_j(i), 1 \le j \le t, 0 \le i \le l\}$, which is a neighborhood of ψ. In the future, we shall make such approximation claims without formulating the accompanying neighborhood specifically.) That is, $\phi\theta\gamma_j(b+i) = \theta\gamma_j(i)$, $1 \le j \le t$, $0 \le i \le l$, a contradiction.

(b) For every $n \in \mathbf{N}_0$ there exists $k_n \in \mathbf{N}$ such that $\gamma_j(k_n+i) = T^{k_n}\gamma_j(i) = \theta\gamma_j(i)$, $1 \le j \le t$, $0 \le i \le n+l$. Therefore, if $0 \le b \le n$ and $b \in B_l$, that is, if $\theta\gamma_j(b+i) = \theta\gamma_j(i)$,

$0 \le i \le l$, $1 \le j \le t$, then $\gamma_j(k_n + b + i) = \theta\gamma_j(b + i) = \theta\gamma_j(i)$, $0 \le i \le l$, $1 \le j \le t$ as well, so that $k_n + b \in P_l$. Therefore,

$$\bigcup_{n=0}^{\infty} \Big(\big(B_l \cap \{0, 1, \cdots, n\}\big) + k_n \Big) \subset P_l.$$

Hence P_l contains a shifted copy of every finite subset of a syndetic set, and is therefore piecewise syndetic by Proposition 1.1.

□

Theorem 1.5 Suppose $J \subset S$ is a minimal left ideal, $\theta \in J$, $(n_\alpha)_{\alpha\in\mathcal{F}}$ is an IP-set in $\mathbf{Z}$ and $k \in \mathbf{N}$. Suppose that $p_1(x), \cdots, p_k(x) \in \mathbf{Z}[x]$ with $p_m(0) = 0$, $1 \le m \le k$. Let $E \subset S^k$ consist of all k-tuples $\big(T^{a+p_1(n_\alpha)}, T^{a+p_2(n_\alpha)}, \cdots, T^{a+p_k(n_\alpha)}\big)$, where $a \in \mathbf{N}$ and $\alpha \in \mathcal{F}$ are such that $a + p_m(n_\alpha) \in \mathbf{N}$, $1 \le m \le k$. Then $(\theta, \cdots, \theta) \in \overline{E}$.

Proof. We must show that for every $t \in \mathbf{N}$ and every choice of $\gamma_1, \cdots, \gamma_t \in X$ and $n_1, \cdots, n_t \in \mathbf{N}_0$ there exists $a \in \mathbf{Z}$ and $\alpha \in \mathcal{F}$ with $a + p_m(n_\alpha) \in \mathbf{N}$, $1 \le m \le k$, such that

$$T^{a+p_m(n_\alpha)}\gamma_j(n_j) = \theta\gamma_j(n_j),\ 1 \le m \le k,\ 1 \le j \le t. \tag{1.1}$$

Let $l = \max\{n_j : 1 \le j \le t\}$. By Proposition 1.4, the set $P_l = \big\{n : \gamma_j(n + i) = \theta\gamma_j(i),\ 1 \le j \le t,\ 0 \le i \le l\big\}$ is piecewise syndetic. Let $\mathcal{A}$ consist of all sets lying in $\mathbf{N}$ and having the form $\big\{a + p_1(n_\alpha), \cdots, a + p_k(n_\alpha)\big\}$ (for some $a \in \mathbf{N}$ and $\alpha \in \mathcal{F}$). According to the Bergelson-Leibman theorem (Theorem C in the introduction), $\mathcal{A}$ is a partition regular family. Therefore, by Theorem 1.2, P_l contains a configuration in $\mathcal{A}$, say $\big\{a + p_1(n_\alpha), \cdots, a + p_k(n_\alpha)\big\}$. It follows that

$$T^{a+p_m(n_\alpha)}\gamma_j(i) = \gamma_j\big(a + p_m(n_\alpha) + i\big) = \theta\gamma_j(i),\ 1 \le j \le t,\ 0 \le i \le l,\ 1 \le m \le k.$$

In particular, (1.1) holds.

□

We are now ready to apply Theorem 1.5 to obtain the proof of (actually something more general than) Theorem A from the introduction. To illustrate the technique, let us first establish a special case; namely let us see that, given a coloring $\mathbf{N} = \bigcup_{i=1}^{r} C_i$, we can find j, with $1 \le j \le r$, and two IP-sets, $(a_\alpha)_{\alpha\in\mathcal{F}}$ and $(m_\alpha)_{\alpha\in\mathcal{F}}$, such that for all $\alpha \in \mathcal{F}$, $\{a_\alpha, a_\alpha + m_\alpha^2\} \subset C_j$.

We begin by letting X, T and S be as above Let θ be a minimal idempotent in S. There exists a unique j with $1 \le j \le r$ such that $\theta \in \overline{\{T^n : n \in C_j\}}$. Put $\gamma = 1_{C_j} \in X$. Clearly $\theta\gamma(0) = 1$. By Theorem 1.5, there exist a_1 and n_1 in $\mathbf{N}$ (Theorem 1.5 says that n_1 can come from any prescribed IP-set, but we won't use this full strength yet) such that $(T^{a_1}, T^{a_1+n_1^2})$ approximates (θ, θ) in S_2 to the extent that

$$\begin{aligned}
&\gamma(a_1) = T^{a_1}\gamma(0) = \theta\gamma(0) = 1,\\
&\theta\gamma(a_1) = T^{a_1}\theta\gamma(0) = \theta^2\gamma(0) = \theta\gamma(0) = 1,\\
&\gamma(a_1 + n_1^2) = T^{a_1+n_1^2}\gamma(0) = \theta\gamma(0) = 1,\ \text{and}\\
&\theta\gamma(a_1 + n_1^2) = T^{a_1+n_1^2}\theta\gamma(0) = \theta^2\gamma(0) = \theta\gamma(0) = 1.
\end{aligned}$$

Next, pick by Theorem 1.5 a_2 and n_2 so that $(T^{a_2}, T^{a_2+n_2^2}, T^{a_2+n_2^2+2n_1n_2})$ approximates (θ, θ, θ) in S^3 to the extent that

$$\begin{aligned}
&\gamma(a_2) = T^{a_2}\gamma(0) = \theta\gamma(0) = 1,\\
&\gamma(a_1 + a_2) = T^{a_2}\gamma(a_1) = \theta\gamma(a_1) = \theta\gamma(0) = 1,\\
&\gamma(a_2 + n_2^2) = T^{a_2+n_2^2}\gamma(0) = \theta\gamma(0) = 1, \text{ and}\\
&\gamma(a_1 + a_2 + (n_1 + n_2)^2) = T^{a_2+n_2^2+2n_1n_2}\gamma(a_1 + n_1^2) = \theta\gamma(a_1 + n_1^2) = \theta\gamma(0) = 1,
\end{aligned}$$

and so that, similarly,

$$\theta\gamma(a_2) = \theta\gamma(a_1 + a_2) = \theta\gamma(a_2 + n_2^2) = \theta\gamma(a_1 + a_2 + (n_1 + n_2)^2) = 1.$$

Choose a_3, n_3 so that $(T^{a_3}, T^{a_3+n_3^2}, T^{a_3+n_3^2+2n_2n_3}, T^{a_3+n_3^2+2n_1n_3}, T^{a_3+n_3^2+2(n_1+n_2)n_3})$ approximates $(\theta, \theta, \theta, \theta, \theta)$ in S^5 to the extent that

$$\begin{aligned}
&\gamma(a_3) = \gamma(a_3 + a_2) = \gamma(a_3 + a_1) = \gamma(a_3 + a_2 + a_1) = \gamma(a_3 + n_3^2)\\
=&\gamma(a_3 + a_2 + (n_3 + n_2)^2) = \gamma(a_3 + a_1 + (n_3 + n_1)^2) = \gamma(a_3 + a_2 + a_1 + (n_3 + n_2 + n_1)^2)\\
=&\theta\gamma(a_3) = \theta\gamma(a_3 + a_2) = \theta\gamma(a_3 + a_1) = \theta\gamma(a_3 + a_2 + a_1) = \theta\gamma(a_3 + n_3^2)\\
=&\theta\gamma(a_3 + a_2 + (n_3 + n_2)^2) = \theta\gamma(a_3 + a_1 + (n_3 + n_1)^2)\\
=&\theta\gamma(a_3 + a_2 + a_1 + (n_3 + n_2 + n_1)^2) = \theta\gamma(0) = 1.
\end{aligned}$$

Continuing in this fashion we achieve our aim, letting $a_\alpha = \sum_{i\in\alpha} a_i$ and $m_\alpha = \sum_{i\in\alpha} n_i$. □

Notice that what is being exploited here is that for any fixed m, the polynomial $q(x) = (x+m)^2 - m^2 = x^2 + 2mx$ has zero constant term. It is a trivial matter that a similar fact holds for any polynomial $p(x) \in \mathbf{Z}[x]$. That is, if m is fixed then $q(x) = p(x+m) - p(m)$ has zero constant term, i.e. $q(0) = 0$. More details for the inductive scheme are given in the proof to follow (albeit in a very general form).

Theorem 1.6 If $r \in \mathbf{N}$, $\mathbf{N} = \bigcup_{i=1}^{r} C_i$, and $(n_\alpha)_{\alpha\in\mathcal{F}}$ is an IP-set in $\mathbf{Z}$ then there exists j, with $1 \le j \le r$, an IP-ring $\mathcal{F}^{(1)}$, and an IP-set $(a_\alpha)_{\alpha\in\mathcal{F}^{(1)}}$ having the property that for every polynomial $p(x) \in \mathbf{Z}[x]$ with $p(0) = 0$, there exists $\beta \in \mathcal{F}$ such that for all $\alpha \in \mathcal{F}^{(1)}$ with $\alpha > \beta$ we have $\big(a_\alpha + p(n_\alpha)\big) \in C_j$.

Proof. Let $p_1(x), p_2(x), \cdots$ be an enumeration of the members of $\mathbf{Z}[x]$ which have zero constant term. Let X, T, and S be defined as above and let θ be any minimal idempotent in S. Pick j with $1 \le j \le r$ such that $\theta \in \overline{\{T^n : n \in J\}}$. Let $\gamma = 1_{C_j}$. Plainly $\theta\gamma(0) = 1$.

By Theorem 1.5, we may select $a_1 \in \mathbf{N}$ and $\alpha_1 \in \mathcal{F}$ such that $(T^{a_1}, T^{a_1+p_1(n_{\alpha_1})})$ approximates (θ, θ) in S^2 to the extent that

$$\begin{aligned}
&\gamma(a_1) = T^{a_1}\gamma(0) = \theta\gamma(0) = 1,\\
&\gamma\big(a_1 + p_1(n_{\alpha_1})\big) = T^{a_1+p_1(n_{\alpha_1})}\gamma(0) = \theta\gamma(0) = 1,\\
&\theta\gamma(a_1) = T^{a_1}\theta\gamma(0) = \theta^2\gamma(0) = \theta\gamma(0) = 1 \text{ and}\\
&\theta\gamma\big(a_1 + p_1(n_{\alpha_1})\big) = T^{a_1+p_1(n_{\alpha_1})}\theta\gamma(0) = \theta^2\gamma(0) = \theta\gamma(0) = 1.
\end{aligned}$$

We introduce some useful notation. If B is a family of sets, we denote by $FU_\emptyset(B)$ the set of finite unions of members of B, including $\emptyset$. Moreover, we agree that $n_\emptyset = 0$. Suppose now that $a_1, \cdots, a_{t-1}$ and $\alpha_1 < \cdots < \alpha_{t-1}$ have been chosen such that

$$\begin{aligned} &\text{for all } j,\ 1 \le j \le t-1, \text{ and all } \alpha \in FU_\emptyset\big(\{\alpha_j, \alpha_{j+1}, \cdots, \alpha_{t-1}\}\big), \\ &\gamma\big(a_\alpha + p(n_\alpha)\big) = 1 \text{ and } \theta\gamma\big(a_\alpha + p(n_\alpha)\big) = 1, \end{aligned} \tag{1.2}$$

where we are writing $a_{\alpha_{i_1} \cup \cdots \cup \alpha_{i_m}} = a_{i_1} + \cdots + a_{i_m}$.

Let $\{q_1(x), q_2(x), \cdots, q_r(x)\}$ consist of all polynomials of the form $q(x) = p_j(x+n_\alpha) - p_j(n_\alpha)$, where $1 \le j \le t$ and $\alpha \in FU_\emptyset\big(\{j, j+1, \cdots, t-1\}\big)$. Note that $q_i(0) = 0$, $1 \le i \le r$. Choose, by Theorem 1.5, $a_t \in \mathbf{N}$ and α_t in $\mathcal{F}$, with $\alpha_t > \alpha_{t-1}$, such that

$$\big(T^{a_t + q_1(n_{\alpha_t})}, T^{a_t + q_2(n_{\alpha_t})}, \cdots, T^{a_t + q_r(n_{\alpha_t})}\big) \sim (\theta, \theta, \cdots, \theta) \text{ in } S^r,$$

where $\sim$ is taken to mean that

$$\begin{aligned} \gamma\big(a_t + a_\alpha + p_j(n_{\alpha_t} + n_\alpha)\big) &= T^{a_t + p_j(n_{\alpha_t} + n_\alpha) - p_j(n_\alpha)}\gamma\big(a_\alpha + p_j(n_\alpha)\big) \\ &= \theta\gamma\big(a_\alpha + p_j(n_\alpha)\big) = \theta\gamma(0) = 1 \text{ and} \\ \theta\gamma\big(a_t + a_\alpha + p_j(n_{\alpha_t} + n_\alpha)\big) &= T^{a_t + p_j(n_{\alpha_t} + n_\alpha) - p(n_\alpha)}\theta\gamma\big(a_\alpha + p_j(n_\alpha)\big) \\ &= \theta^2\gamma\big(a_\alpha + p_j(n_\alpha)\big) = \theta\gamma\big(a_\alpha + p_j(n_\alpha)\big) = \theta\gamma(0) = 1 \end{aligned}$$

for all j, $1 \le j \le t$, and all $\alpha \in FU_\emptyset\big(\{\alpha_j, \alpha_{j+1}, \cdots, \alpha_{t-1}\}\big)$. One may now check that (1.2) holds with $t-1$ replaced by t. Continue until $(A_i)_{i=1}^\infty$ and $(\alpha_i)_{i=1}^\infty$ have been chosen and let $\mathcal{F}^{(1)} = FU\big((\alpha_i)_{i=1}^\infty\big)$. We have $\big(a_\alpha + p_i(n_\alpha)\big) \in C_j$ for all $\alpha \in \mathcal{F}^{(1)}$ with $\alpha > \alpha_i$, completing the proof.

□

2. Shift invariant configurations in semigroups.

In this section, G will be any semigroup. Let $G_e = G \cup \{e\}$, where e is an identity for G (not necessarily an element of G). For $r \in \mathbf{N}$, put $X = \{0,1\}^{G_e}$. With the product topology, X is compact. We consider the space $\Omega = X^X$, with the product topology, which is a compact right topological semigroup under composition, and embed G in Ω by putting $T_g\gamma(h) = \gamma(hg)$ for $\gamma \in X$ and $h \in G$. Let

$$S = \overline{\{T_g : g \in G\}}.$$

As we pointed out in the previous section (for $G = \mathbf{N}$), for every $A, B \subset G$ we have $(\overline{A})(\overline{B}) \subset \overline{AB}$. In particular, S is a semigroup.

The following general version of Proposition 1.4 is again well known. Again for completeness, we give a proof.

Proposition 2.1. Let $J \subset S$ be a minimal left ideal, let $\theta \in J$, $t \in \mathbf{N}$ and $\gamma_1, \cdots, \gamma_t \in X$. If $F \subset G$ is a finite set then:

(a) the set $B_F = \{g : \theta\gamma_j(hg) = \theta\gamma_j(h),\ 1 \le j \le t,\ h \in F\}$ is left syndetic.

(b) the set $P_F = \{g : \gamma_j(hg) = \theta\gamma_j(h),\ 1 \le j \le t,\ h \in F\}$ is left piecewise syndetic.

Proof. (a) Suppose not. That is, assume that B_F^c is left thick. Then for every finite set $H \subset G$, there exists $g_H \in G$ such that $Hg_H \subset B_F^c$. $\{T_{g_H}\}$ is a net in S indexed by the family of finite subsets of G. Choose any convergent subnet and let ϕ be its limit. Then for every $b \in G$, we can conclude (by approximating ϕ by some T_{g_H}, where $b \in H$) that $\phi\theta\gamma_j(hb) \neq \theta\gamma_j(h)$ for some j, $1 \le j \le t$, and some $h \in F$. Since J is a minimal left ideal, there exists $\psi \in S$ such that $\theta = \psi\phi\theta$. Hence we many choose $b \in G$ close enough to ψ that $T_b\phi\theta\gamma_j(h) = \psi\phi\theta\gamma_j(h) = \theta\gamma(h)$, $1 \le j \le t$, $h \in F$. That is, $\phi\theta\gamma_j(hb) = \theta\gamma_j(h)$ for all j, $1 \le j \le t$, and all $h \in F$. This is a contradiction.

(b) For every $n \in \mathbf{N}$ there exists $k_n \in G$ such that

$$\gamma_j(hbk_n) = T_{k_n}\gamma_j(hb) = \theta\gamma_j(hb),\ 1 \le j \le t,\ h \in F,\ b \in E_n.$$

It follows that for all $b \in (E_n \cap B_F)$, we have $\gamma_j(hbk_n) = \theta\gamma_j(hb) = \theta\gamma_j(h)$, $1 \le j \le t$, $h \in F$, implying that $bk_n \in P_F$. Therefore

$$\bigcup_{n=1}^{\infty} \big((E_n \cap B_F)k_n\big) \subset P_F.$$

By Proposition 1.1, P_F is left piecewise syndetic.

□

The following theorem serves the same function as Theorem 1.5 in the previous section.

Theorem 2.2. Let $J \subset S$ be a minimal left ideal and let $\theta \in J$. Let $\mathcal{A}$ be a two sided shift invariant, partition regular family of finite subsets of G. Let $t \in \mathbf{N}$, $\gamma_1, \cdots, \gamma_t \in X$ and $h_1, \cdots, h_t \in G$. There exists $A \in \mathcal{A}$ such that

$$\gamma_j(h_ja) = T_a\gamma_j(h_j) = \theta\gamma_j(h_j),\ a \in A,\ 1 \le j \le t.$$

Proof. By Proposition 2.1 the set $P = \{g : \gamma_j(h_jg) = \theta\gamma_j(h_j),\ 1 \le j \le t\}$ is left piecewise syndetic. Therefore by Theorem 1.2 P contains some $A \in \mathcal{A}$.

□

As was stated earlier, the following theorem may be used to get Theorem B from the introduction (provided one assumes the finitary version of Theorem B).

Theorem 2.3. Let G be a semigroup, and suppose that $\mathcal{A}$ is a two-sided shift invariant, partition regular family of finite subsets of G. Let $r \in \mathbf{N}$. For any partition $G = \bigcup_{i=1}^r C_i$, there exists j with $1 \le j \le r$ and a sequence $(A_n)_{n=1}^\infty \subset \mathcal{A}$ such that the set of all finite products of the form $a_{n_1}a_{n_2}\cdots a_{n_m}$, where $n_1 < \cdots < n_m$ and $a_{n_i} \in A_{n_i}$, is contained in C_j.

Proof. Let θ be any minimal idempotent in S. There exists j with $1 \le j \le r$ such that $\theta \in \overline{\{T_g : g \in C_j\}}$. Putting $\gamma = 1_{C_j}$, we have $\theta\gamma(e) = 1$. By Theorem 2.2, we may select $A_1 \in \mathcal{A}$ such that, for all $a \in A_1$,

$$\gamma(a) = \theta\gamma(e) \text{ and } \theta\gamma(a) = \theta^2\gamma(e) = \theta\gamma(e) = 1.$$

Having chosen $A_1, \cdots, A_{t-1}$ such that

$$\begin{aligned} &\gamma(a_{n_1} \cdots a_{n_m}) = 1 \text{ and } \theta\gamma(a_{n_1} \cdots a_{n_m}) = 1 \\ &\text{for all } 1 \leq n_1 < \cdots < n_m \leq t-1 \text{ and } a_{n_i} \in A_{n_i},\ 1 \leq i \leq m, \end{aligned} \tag{2.1}$$

select, by Theorem 2.2, $A_t \in \mathcal{A}$ such that for all $a_t \in A_t$,

$$\begin{aligned} &\gamma(a_{n_1} \cdots a_{n_m} a_t) = \theta\gamma(a_{n_1} \cdots a_{n_m}) = \theta\gamma(e) = 1 \text{ and} \\ &\theta\gamma(a_{n_1} \cdots a_{n_m} a_t) = \theta^2\gamma(a_{n_1} \cdots a_{n_m}) = \theta\gamma(a_{n_1} \cdots a_{n_m}) = \theta\gamma(e) = 1 \\ &\text{for all } 1 \leq n_1 < \cdots < n_m \leq t-1 \text{ and } a_{n_i} \in A_{n_i},\ 1 \leq i \leq m. \end{aligned}$$

It follows that (2.1) holds with $t-1$ replaced by t. Continuing in this manner completes the proof.

□

Theorem 2.3 has many natural applications. For example, suppose G is taken to be $\mathbf{N}$ and $\mathcal{A}$ is taken to be the family of singletons. Theorem 2.3 in this context is Hindman's theorem. If $\mathcal{A} = \big\{\{a_1, \cdots, a_k\} : (a_1, \cdots, a_k) \in I\big\}$, where I is a two-sided ideal in some semigroup of G^k which contains the diagonal, then Theorem 2.3 in this context implies Theorem B from the introduction. (Notice that in order to apply Theorem 2.3 in this case we need to know that $\mathcal{A}$ is partition regular, which is a consequence of the finitary version of Theorem B.)

For some applications, Theorem 2.3 isn't quite what we need. Theorem 1.6, for example, is not an immediate consequence of Theorem 2.3. One can see this already in the special case (which dealt with the single polynomial $p(n) = n^2$) treated after Theorem 1.5. At the first stage of the inductive procedure, the family under consideration was

$$\mathcal{A}_1 = \big\{\{a, a+n^2\} :\ a, n \in \mathbf{N}\big\}.$$

At the next stage, it was

$$\mathcal{A}_2 = \big\{\{a, a+n^2, a+n^2+2n_1n\} :\ a, n \in \mathbf{N}\big\}.$$

After that it was

$$\mathcal{A}_3 = \big\{\{a, a+n^2, a+n^2+2n_1n, a+n^2+2n_2n, a+n^2+2(n_1+n_2)n\} :\ a, n \in \mathbf{N}\big\}.$$

Notice that at each stage, the family one needed to consider depended on previous choices of n_i. Taking this into account, we formulate the following more general version of Theorem 2.3, which can be proved in the same way.

Theorem 2.4 Let G be a semigroup, let S be a set, and suppose that for every $s \in S$, T_s is a set. For each $s \in S$, let $\mathcal{A}_s$ be a 2-sided shift invariant, partition regular family of finite subsets of G indexed by T_s; namely, for each $t \in T_s$, let $A_{s,t} \in \mathcal{A}_s$, such that $\mathcal{A}_s = \{A_{s,t} : t \in T_s\}$. Let $s_1 \in S$ and suppose $\phi : \{(s,t) : s \in S, t \in T_s\} \to S$ is a function. For each finite coloring of G there exist sequences $(s_n)_{n=2}^{\infty} \subset S$ and $(t_n)_{n=1}^{\infty}$ such that $t_n \in T_{s_{n-1}}$ and such that $\phi(s_{n-1}, t_{n-1}) = s_n$ for $n \geq 2$ and such that the set of all products $a_{n_1} a_{n_2} \cdots a_{n_m}$, where $n_1 < \cdots < n_m$ and $a_{n_i} \in A_{s_{n_i}, t_{n_i}}$, is monochromatic.

We now move to an application of Theorem 2.4 involving *VIP-systems*, which were introduced in [BFM]. VIP-systems are variants of IP-sets having a "polynomial" nature. In a commutative group $(G, +)$, a sequence indexed by $\mathcal{F}$ (more generally, by any IP-ring $\mathcal{F}^{(1)}$), say $(v_\alpha)_{\alpha \in \mathcal{F}}$, is called a VIP-system if there exists some non-negative integer d such that for every pairwise disjoint $\alpha_0, \alpha_1, \cdots, \alpha_d \in \mathcal{F}$ we have

$$\sum_{\substack{\{\beta_1, \cdots, \beta_t\} \subset \{\alpha_0, \cdots, \alpha_d\} \\ \beta_i \neq \beta_j,\ 1 \leq i < j \leq t}} (-1)^t v_{\beta_1 \cup \cdots \cup \beta_t} = 0. \tag{2.2}$$

If $(v_\alpha)_{\alpha \in \mathcal{F}}$ is a VIP-system then the least non-negative d for which (2.2) holds is called the *degree* of the system. If $d = 1$, this equation reduces to simply $v_{\alpha_0 \cup \alpha_1} = v_{\alpha_0} + v_{\alpha_1}$. In other words, the VIP-systems of degree 1 are just the (non-identically zero) IP-sets.

Recall the alternative characterization of IP-sets; namely as the set of finite sums of some sequence. A similar characterization of VIP-systems will be useful. For $d \in \mathbf{N}$, let $\mathcal{F}_d$ denote the family of non-empty subsets of $\mathbf{N}$ having cardinality at most d.

Proposition 2.5 Let G be an additive abelian group and let $d \in \mathbf{N}$. A sequence indexed by $\mathcal{F}$, $(v_\alpha)_{\alpha \in \mathcal{F}}$, in G is a VIP-system of degree at most d if and only if there exists a function from $\mathcal{F}_d$ to G, written $\gamma \to n_\gamma$, $\gamma \in \mathcal{F}_d$, such that

$$v_\alpha = \sum_{\gamma \subset \alpha, \gamma \in \mathcal{F}_d} n_\gamma \tag{2.3}$$

for all $\alpha \in \mathcal{F}$.

Proof. First we establish that any sequence $(v_\alpha)_{\alpha \in \mathcal{F}}$ satisfying (2.3) is a VIP-system of degree at most d. Let $\alpha_0, \alpha_1, \cdots, \alpha_d$ be pairwise disjoint members of $\mathcal{F}$. We need to show that

$$\sum_{\substack{\{\beta_1, \cdots, \beta_t\} \subset \{\alpha_0, \cdots, \alpha_d\} \\ \beta_i \neq \beta_j,\ 1 \leq i < j \leq t}} (-1)^t \Big(\sum_{\gamma \subset (\beta_1 \cup \cdots \cup \beta_t), \gamma \in \mathcal{F}_d} n_\gamma \Big) = 0. \tag{2.4}$$

Fix $\gamma \in \mathcal{F}_d$ with $\gamma \subset (\alpha_0 \cup \alpha_1 \cup \cdots \cup \alpha_d)$. Let k be the number of α_i's γ intersects non-trivially. For $1 \leq t \leq d$, the number of choices of pairwise distinct $\beta_1, \cdots, \beta_t$ from $\{\alpha_0, \cdots, \alpha_d\}$ such that $\gamma \subset (\beta_1 \cup \cdots \cup \beta_t)$ is 0 if $t < k$ and $\binom{d+1-k}{t-k}$ otherwise. Hence the number of times n_γ is counted in (2.4) is $\sum_{i=0}^{d+1-k} \binom{d+1-k}{i} (-1)^i = (1-1)^{d+1-k} = 0$. Since γ was arbitrary we are done.

For the converse, suppose $(v_\alpha)_{\alpha \in \mathcal{F}}$ is a VIP-system of degree d. For each $\gamma \in \mathcal{F}_d$, put

$$n_\gamma = \sum_{\beta \subset \gamma} (-1)^{|\gamma| - |\beta|} v_\beta.$$

We need to show that

$$v_\alpha = \sum_{\gamma \subset \alpha, \gamma \in \mathcal{F}_d} n_\gamma \tag{2.5}$$

for all $\alpha \in \mathcal{F}$.

If $|\alpha| \leq d$ this reduces to

$$v_\alpha = \sum_{\gamma\subset\alpha}\sum_{\beta\subset\gamma}(-1)^{|\gamma|-|\beta|}v_\beta. \tag{2.6}$$

Let $\beta \subset \alpha$. We shall count how many times v_β is counted on the right hand side of (2.6). If $\beta = \alpha$, clearly once. Otherwise, for $|\beta| \leq k \leq |\alpha|$ there are $\binom{|\alpha|-|\beta|}{k-|\beta|}$ sets γ with $\beta \subset \gamma \subset \alpha$ and $|\gamma| = k$. Hence β is counted $\sum_{k=|\beta|}^{|\alpha|}\binom{|\alpha|-|\beta|}{k-|\beta|}(-1)^{k-|\beta|} = \sum_{i=0}^{|\alpha|-|\beta|}\binom{|\alpha|-|\beta|}{i}(-1)^i = 0$ times, yielding (2.5) as required.

For $|\alpha| > d$, we use induction. Let $\alpha \in \mathcal{F}$ with $|\alpha| > d$ and suppose (2.5) holds for all sets of cardinality less than $|\alpha|$. Write $\alpha = \alpha_0 \cup \alpha_1 \cup \cdots \cup \alpha_d$, where $\alpha_i \cap \alpha_j = \emptyset$ for $0 \leq i \neq j \leq d$. We have

$$\sum_{\substack{\{\beta_1,\cdots,\beta_t\}\subset\{\alpha_0,\cdots,\alpha_d\}\\ \beta_i\neq\beta_j,\ 1\leq i<j\leq t}} (-1)^t v_{\beta_1\cup\cdots\cup\beta_t} = 0. \tag{2.7}$$

On the other hand, by the prior implication (2.4) holds, namely

$$\sum_{\substack{\{\beta_1,\cdots,\beta_t\}\subset\{\alpha_0,\cdots,\alpha_d\}\\ \beta_i\neq\beta_j,\ 1\leq i<j\leq t}} (-1)^t\Big(\sum_{\gamma\subset(\beta_1\cup\cdots\cup\beta_t),\gamma\in\mathcal{F}_d} n_\gamma\Big) = 0. \tag{2.8}$$

Since, according to the induction hypothesis, $v_{\beta_1\cup\cdots\cup\beta_t} = \sum_{\gamma\subset(\beta_1\cup\cdots\cup\beta_t),\gamma\in\mathcal{F}_d} n_\gamma$ when $\{\beta_1,\cdots,\beta_t\}$ is a proper subset of $\{\alpha_0,\cdots,\alpha_d\}$, equations (2.7) and (2.8) establish it for $\{\beta_1,\cdots,\beta_t\} = \{\alpha_0,\cdots,\alpha_d\}$, as desired.

□

Many natural polynomially generated sequences $(v_\alpha)_{\alpha\in\mathcal{F}}$ can be shown to be VIP-systems. For example, let $(n_\alpha)_{\alpha\in\mathcal{F}}$ be an IP-set in $\mathbf{Z}$, and put $v_\alpha = n_\alpha^2$. Then $(v_\alpha)_{\alpha\in\mathcal{F}}$ is a VIP-system of degree (at most) 2, for letting $a_i = n_{\{i\}}^2$ and $b_{ij} = 2n_{\{i\}}n_{\{j\}}$, one sees with the help of the multinomial theorem that (2.3) holds. This phenomenon is fully elucidated by the following proposition.

Proposition 2.6 Let R be a commutative ring, $k, d \in \mathbf{N}$, and let $p \in R[x_1, x_2, \cdots, x_k]$ be a polynomial of degree d with coefficients in R and with $p(0, \cdots, 0) = 0$. If $(n_\alpha^{(i)})_{\alpha\in\mathcal{F}}$ are IP-sets in R, $1 \leq i \leq k$, then letting $v_\alpha = p(n_\alpha^{(1)}, n_\alpha^{(2)}, \cdots, n_\alpha^{(k)})$, $\alpha \in \mathcal{F}$, the resulting sequence $(v_\alpha)_{\alpha\in\mathcal{F}}$ is a VIP-system of degree at most d.

Proof. It suffices to establish it for $k = d$ and monomials of the form $p(x_1, \cdots, x_d) = gx_1x_2\cdots x_d$, where $g \in R$ (The reason we may assume $k = d$ and that all the exponents are 1 is that some of the IP-sets can be repeats. The reason we may assume p is a monomial is that the sum of two VIP-systems of degree at most d is again such.)

We use the characterization of Proposition 2.5. Namely, let, for $\gamma \in \mathcal{F}_d$,

$$n_\gamma = \sum_{\{a_1,a_2,\cdots,a_d\}=\gamma} g n^{(1)}_{\{a_1\}} \cdots n^{(d)}_{\{a_d\}}.$$

One may easily check that

$$v_\alpha = \sum_{\gamma\subset\alpha,\gamma\in\mathcal{F}_d} n_\gamma$$

for all $\alpha \in \mathcal{F}$.

□

Our concern with VIP-systems arises out of the following theorem. It is a consequence of a polynomial Hales-Jewett theorem ([BL2]) due to Bergelson and Leibman. (We remark, however, that a direct proof of Theorem 2.7 could be given by the more elementary methods of [BL1].)

Theorem 2.7 Let G be an additive abelian group and let $k \in \mathbf{N}$. If $(v^{(i)}_\alpha)_{\alpha\in\mathcal{F}}$ are VIP-systems in G, $1 \le i \le k$, then for any finite coloring of G there exists a monochromatic configuration of the form

$$\{a + v^{(1)}_\alpha, a + v^{(2)}_\alpha, \cdots, a + v^{(k)}_\alpha\},$$

where $a \in G$ and $\alpha \in \mathcal{F}$.

As an application of Theorem 2.4, we shall give an infinitary version of Theorem 2.7. Recall that in proving Theorem 1.6 we used the fact that for a polynomial $p(x)$ and fixed integer m, the polynomial $q(x) = p(x+m) - p(m)$ has zero constant term. We make here a similar observation, namely that if $(v_\alpha)_{\alpha\in\mathcal{F}}$ is a VIP-system (of degree d) in a commutative group G, and $\beta \in \mathcal{F}$ is fixed, then letting $u^{(\beta)}_\alpha = v_{\alpha\cup\beta} - v_\beta$ for all $\alpha \in f$ satisfying $\alpha\cap\beta = \emptyset$, the sequence $(u^{(\beta)}_\alpha)_{\alpha\in\mathcal{F},\alpha\cap\beta=\emptyset}$ is a VIP-system (indeed, of degree at most d).

This is easily seen via the characterization of Proposition 2.5. Namely, there exists a sequence $(n_\gamma)_{\gamma\in\mathcal{F}_d}$ such that $v_\alpha = \sum_{\gamma\subset\alpha,\gamma\in\mathcal{F}_d} n_\gamma$. We have

$$u^{(\beta)}_\alpha = \sum_{\gamma\subset\alpha\cup\beta,\gamma\in\mathcal{F}_d} n_\gamma - \sum_{\gamma\subset\beta,\gamma\in\mathcal{F}_d} n_\gamma = \sum_{\gamma\subset\alpha,\gamma\in\mathcal{F}_d} m_\gamma,$$

where $m_\gamma = \sum_{\xi\subset\beta,\xi\cup\gamma\in\mathcal{F}_d} n_{\xi\cup\gamma}$.

Theorem 2.8 Let G be an additive abelian group and let

$$(v_\alpha)_{\alpha\in\mathcal{F}}, (w_\alpha)_{\alpha\in\mathcal{F}}, \cdots, (z_\alpha)_{\alpha\in\mathcal{F}}$$

be VIP-systems in G. For any $r \in \mathbf{N}$ and any r-coloring $G = \bigcup_{i=1}^r C_i$ there exists j, $1 \le j \le r$, an IP-ring $\mathcal{F}^{(1)}$, and an IP-set $(a_\alpha)_{\alpha\in\mathcal{F}^{(1)}}$ such that for all $\alpha \in \mathcal{F}^{(1)}$ we have

$$\{a_\alpha + v_\alpha, a_\alpha + w_\alpha, \cdots, a_\alpha + z_\alpha\} \subset C_j.$$

Proof. We use Theorem 2.4. Let

$$S = \bigcup_{n=1}^{\infty} \{(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) : \alpha_1, \alpha_2, \cdots, \alpha_{n-1} \in \mathcal{F},\ \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1}\}$$

be the set of finite, increasing sequences taken from $\mathcal{F}$ (by considering $n = 1$ in the union, we are including the empty sequence). For $s = (\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) \in S$, let

$$\begin{aligned}\mathcal{A}_s &= \mathcal{A}_{(\alpha_1,\alpha_2,\cdots,\alpha_{n-1})} \\ &= \Big\{ \bigcup_{\beta \in FU_\emptyset(\alpha_1,\alpha_2,\cdots,\alpha_{n-1})} \{a + v_\alpha^{(\beta)}, a + w_\alpha^{(\beta)}, \cdots, a + z_\alpha^{(\beta)}\} : \alpha \in \mathcal{F},\ \alpha > \alpha_{n-1},\ a \in G \Big\}\end{aligned}$$

and let $T_s = G \times \{\alpha \in \mathcal{F} : \alpha > \alpha_{n-1}\}$.

By Theorem 2.7 and the fact that $(v_\alpha^{(\beta)})_{\alpha>\alpha_{n-1}}$, $(w_\alpha^{(\beta)})_{\alpha>\alpha_{n-1}}, \cdots, (z_\alpha^{(\beta)})_{\alpha>\alpha_{n-1}}$ are VIP-systems for fixed $\beta \in FU_\emptyset(\alpha_1, \cdots, \alpha_{n-1})$, for all $s \in S$ $\mathcal{A}_s$ is a 2-sided shift invariant, partition regular family whose members are indexed by T_s. Namely, we let $A_{s,a,\alpha}$, where $s = (\alpha_1, \cdots, \alpha_{n-1}) \in S$, $a \in G$, and $\alpha \in \mathcal{F}$, with $\alpha > \alpha_{n-1}$, be the set

$$A_{s,a,\alpha} = \bigcup_{\beta \in FU_\emptyset(\alpha_1,\alpha_2,\cdots,\alpha_{n-1})} \{a + v_\alpha^{(\beta)}, a + w_\alpha^{(\beta)}, \cdots, a + z_\alpha^{(\beta)}\}.$$

Also let $\phi(s, a, \alpha) = (\alpha_1, \cdots, \alpha_{n-1}, \alpha) \in S$.

Let s_1 be the empty sequence. Theorem 2.4 guarantees us sequences $(s_n)_{n=2}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ such that, firstly, $\phi(s_{n-1}, t_{n-1}) = s_n$, $n \geq 2$. From this it follows that there exists an increasing sequence $(\alpha_n)_{n=1}^{\infty} \subset \mathcal{F}$ such that $s_n = (\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$, $n \in \mathbf{N}$, and a sequence $(a_n)_{n=1}^{\infty} \subset G$ such that $t_n = (a_n, \alpha_n)$. Put $\mathcal{F}^{(1)} = FU\big((\alpha_n)_{n=1}^{\infty}\big)$. For $\alpha \in \mathcal{F}^{(1)}$, put $a_\alpha = \sum_{\alpha_i \subset \alpha} a_i$. Then $(a_\alpha)_{\alpha \in \mathcal{F}^{(1)}}$ is an IP-set.

Secondly, the set of all sums $b_{n_1} + b_{n_2} + \cdots + b_{n_m}$, where $n_1 < \cdots < n_m$ and $b_{n_i} \in A_{s_{n_i}, t_{n_i}}$, is monochromatic. We claim that, in particular, the set

$$\bigcup_{\alpha \in \mathcal{F}^{(1)}} \{a_\alpha + v_\alpha, a_\alpha + w_\alpha, \cdots, a_\alpha + z_\alpha\}$$

is monochromatic. To see this, let $\alpha = \alpha_{n_1} \cup \cdots \cup \alpha_{n_m}$ be a member of $\mathcal{F}^{(1)}$. We have, for example,

$$a_\alpha + v_\alpha = (a_{n_1} + v_{\alpha_{n_1}}) + (a_{n_2} + v_{\alpha_{n_2}}^{(\alpha_{n_1})}) + (a_{n_3} + v_{\alpha_{n_3}}^{(\alpha_{n_1} \cup \alpha_{n_2})}) + \cdots + (a_{n_m} + v_{\alpha_{n_m}}^{(\alpha_{n_1} \cup \cdots \cup \alpha_{n_{m-1}})}).$$

This is a sum of terms $b_{n_1}, \cdots, b_{n_m}$ with $b_{n_i} \in A_{s_{n_i}, t_{n_i}}$.

□

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