A concentration function estimate and intersective sets from matrices

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Abstract

We give several sufficient conditions on an infinite integer matrix (d_{ij}) for the set $R = \big\{ \sum_{ij\in\alpha,\, i>j} d_{ij} : \alpha \subset \mathbb{N},\, |\alpha| < \infty \big\}$ to be a density intersective set, including the cases $d_{nj} = j^n(1 + O(1/n^{1+\epsilon}))$ and $0 < d_{nj} = o\big(\sqrt{\frac{n}{\log n}}\big)$. For the latter, a concentration function estimate that is of independent interest is applied to sums of sequences of 2-valued random variables whose means may tend to ∞ as $\sqrt{\frac{n}{\log n}}$.

1 Introduction

This paper is concerned with density intersective sets in $\mathbb{Z}$.

Definition. A set $R \subset \mathbb{Z}$ is *density intersective* if for every $A \subset \mathbb{N}$ with $d^*(A) := \limsup_{b-a\to\infty} \frac{|A\cap\{a+1,\dots,b\}|}{b-a} > 0$, one has $R \cap (A - A) \neq \emptyset$.

According to the Furstenberg correspondence principle, R is density intersective if and only if it is a *set of measurable recurrence*, i.e., if for every invertible measure preserving transformation T of a probability space $(X, \mathcal{A}, \mu)$ and every $A \in \mathcal{A}$ with

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$\mu(A) > 0$, there is some $n \in R$ such that $\mu(A \cap T^{-n}A) > 0$; see [F]. Proofs here proceed via the ergodic-theoretic formulation.

We will address some cases of the following conjecture, which is implicit in [BFM].

Conjecture 1.1. *Let $(d_{ij})_{ij\in\mathbb{N}}$ be an infinite matrix with entries from $\mathbb{Z}$. Then $R = \{\sum_{i,j\in\alpha} d_{ij} : \alpha \subset \mathbb{N},\ 0 < |\alpha| < \infty\}$ is density intersective.*

Anecdotal evidence for the truth of the conjecture is provided by the fact that the set R in question is in general *chromatically intersective*, i.e., it meets $\bigcup_{i=1}^{r}(C_i - C_i)$ whenever $\mathbb{N} = \bigcup_{i=1}^{r} C_i$ is a finite partition. This fact follow follows from the more powerful polynomial Hales-Jewett Theorem [BL]; however see Section 1.7 of [Mc1] for a direct proof.

Here are a few cases in which Conjecture 1.1 was previously known to hold:

1. $d_{ij} = 1$. This is Sárközy's theorem [S], which states that the set of square numbers is density intersective.
2. $d_{ij} = \sum_{t=1}^{k} n_i^{(t)} m_j^{(t)}$, where $n_i^{(t)}, m_j^{(t)} \in \mathbb{Z}$ are arbitrary. See [BFM].
3. $d_{ij} = \sum_{t=1}^{k} n_i^{(t)} m_j^{(t)}$ if $i \geq j$, $d_{ij} = 0$ otherwise; where $n_i^{(t)}, m_j^{(t)} \in \mathbb{Z}$, are arbitrary. See [BHåM].

In this paper, we use a mixture of ergodic theory, ultrafilter methods, combinatorial reasoning and harmonic analysis to provide an affirmative answer in several new cases, encompassing those in which $d_{nj} = j^n(1 + O(1/n^{1+\epsilon}))$ and those in which $d_{nj} = o\left(\sqrt{\frac{n}{\log n}}\right)$ as $n \to \infty$ for each fixed j. Higher degree versions of our results are possible, though we confine ourselves here to degree two in order to simplify the exposition.

A distinguishing feature of our results is a greater robustness (insensitivity to perturbation of the matrix (d_{ij})) than in examples 1–3 above. Indeed, rate-of-growth considerations together with mildly restrictive inequalities in the columns of the matrix (d_{ij}) will be used in place of the more constraining equations characterizing 1–3.

2 Ultrafilters on the finite subsets of $\mathbb{N}$

In this section we introduce and elaborate on a recently developed (cf. [B, BM1, BM2]) ultrafilter based methodology for dealing with recurrence questions in ergodic theory. Although this material is somewhat esoteric, our proofs seem to require it.

Definition. If S is a set, we denote by $\mathcal{F}(S)$ the set of non-empty, finite subsets of S. We abbreviate $\mathcal{F}(\mathbb{N})$ by simply $\mathcal{F}$, and for $n \in \mathbb{N}$, write $\mathcal{F}_n = \{\alpha \in \mathcal{F} : \min \alpha > n\} = \mathcal{F}\big(\{n+1, n+2, \dots\}\big)$.

Definition. Let $A \subset \mathcal{F}$. The *upper density* of A is the number

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \mathcal{F}\big(\{1,2,\dots,n\}\big)|}{2^n}.$$

The *lower density* $\underline{d}(A)$ is defined similarly. Note that $\overline{d}(\mathcal{F}_n) = \underline{d}(\mathcal{F}_n) = \frac{1}{2^n}$.

For $\alpha, \beta \in \mathcal{F}$, write $\alpha < \beta$ if $\max \alpha < \min \beta$. If $\alpha < \beta$, write $\alpha * \beta = \alpha \cup \beta$. ($\alpha * \beta$ is undefined otherwise.)

One may check that the pair $(\mathcal{F}, *)$ is an *adequate partial semigroup* in the sense of [BBH] (see also [HM]). Briefly, this means that $*$ maps a subset of $\mathcal{F} \times \mathcal{F}$ to $\mathcal{F}$, is associative for all triples where defined, and for any $\alpha_1, \dots, \alpha_n \in \mathcal{F}$ there is a β such that $\alpha_i * \beta$ is defined for all i, $1 \le i \le n$.

We will be dealing with the Stone-Čech compactification $\beta\mathcal{F}$ of $\mathcal{F}$. We take the points of $\beta\mathcal{F}$ to be ultrafilters on $\mathcal{F}$, the principal ultrafilters being identified with the points of $\mathcal{F}$. Given a set $A \subset \mathcal{F}$, the closure of A is given by $\overline{A} = \{p \in \beta\mathcal{F} : A \in p\}$. The set $\{\overline{A} : A \subset \mathcal{F}\}$ is a basis for the closed (and also the open) sets of $\beta\mathcal{F}$.

For $\alpha \in \mathcal{F}$ and $A \subset \mathcal{F}$, write $\alpha^{-1}A = \{\beta \in \mathcal{F}_{\max \alpha} : \alpha * \beta \in A\}$.

Definition. Let $\delta\mathcal{F} = \bigcap_n \overline{\mathcal{F}_n}$. Now for $p \in \beta\mathcal{F}$ and $q \in \delta\mathcal{F}$, define $p * q \in \beta\mathcal{F}$ by the rule $A \in p * q$ if and only if $\{\alpha \in \mathcal{F} : \alpha^{-1}A \in q\} \in p$.

One can show that this extends $*$ as previous introduced and remains associative where defined. Moreover, $(\delta\mathcal{F}, *)$ is a compact Hausdorff right topological semigroup. (For more information, see [HM, Section 2].)

Any compact Hausdorff right topological semigroup contains an idempotent. An idempotent $p \in \delta\mathcal{F}$ having the property that $\overline{d}(A \cap \mathcal{F}_n) > 0$ for every $A \in p$ and $n \in \mathbb{N}$ is called an *essential* idempotent.

Proposition 2.1. *There exists an essential idempotent $p \in \delta\mathcal{F}$.*

Proof. Let $\mathcal{L} = \{A \subset \mathcal{F} : \exists n \in \mathbb{N} \text{ such that } \underline{d}(A \cap \mathcal{F}_n) = \frac{1}{2^n}\}$. One may show that $\mathcal{L}$ is a filter, and so by Zorn's lemma is contained in some ultrafilter, call it q. As $\mathcal{F}_n \in \mathcal{L} \subset q$ for all n, one has $q \in \bigcap_n \overline{\mathcal{F}_n} = \delta\mathcal{F}$. We claim that for every $A \in q$ and $n \in \mathbb{N}$, one has $\overline{d}(A \cap \mathcal{F}_n) > 0$. For, if $\overline{d}(A \cap \mathcal{F}_n) = 0$, then $\underline{d}(A^c \cap \mathcal{F}_n) = \frac{1}{2^n}$, so that $A^c \in \mathcal{L} \subset q$, and hence $A \notin q$.

Next, note that $\delta\mathcal{F} * q = \{r * q : r \in \delta\mathcal{F}\}$ is a closed left ideal (in particular, a compact Hausdorff right topological semigroup itself) in $\delta\mathcal{F}$, and hence contains an idempotent p. One has $p = r * q$ for some r. If $A \in p = r * q$ and $n \in \mathbb{N}$, then $\{\alpha \in \mathcal{F} : \alpha^{-1}A \in q\} \in r$. In particular, since $r \in \delta\mathcal{F} \subset \overline{\mathcal{F}_n}$, $\mathcal{F}_n \in r$, and so there is some $\alpha \in \mathcal{F}_n$ such that $\alpha^{-1}A \in q$. Since $\alpha^{-1}A \subset \mathcal{F}_n$, $\overline{d}(\alpha^{-1}A) = \overline{d}(\alpha^{-1}A \cap \mathcal{F}_n) > 0$. Also, for all $m > n$ one has

$$|A \cap \mathcal{F}\big(\{n+1, n+2, \dots, m\}\big)| \geq |\alpha^{-1}A \cap \mathcal{F}\big(\{n+1, n+2, \dots, m\}\big)|$$

(the map $\beta \mapsto \alpha * \beta$ from the latter set to the former is injective), hence $\overline{d}(A \cap \mathcal{F}_n) > 0$. □

Let X be a topological space and $f \colon \mathcal{F} \to X$ a function. If $p \in \beta\mathcal{F}$ and $x \in X$, we write $p\text{-}\lim_\alpha f(\alpha) = x$ if for every neighborhood U of x, $\{\alpha : f(\alpha) \in U\} \in p$. Note that if X is compact and Hausdorff, then the p-limit always exists and is unique.

3 $\mathcal{F}$-linear and $\mathcal{F}$-quadratic functions

Throughout this section, G will denote a general countable additive abelian group, though we will consider only $G = \mathbb{Z}$ and the direct sum of countably many copies of $\mathbb{Z}_{k+1}$, which we denote by $G = \bigoplus_{i \in \mathbb{N}} \mathbb{Z}_{k+1}$, in the sequel. (Though we are interested primarily in the integers, some of our constructions are imported from $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}_{k+1}$; proofs for general G are in any case virtually identical.) We will also consider unitary and measure preserving actions of G on Hilbert spaces and probability spaces, respectively. These will be denoted interchangeably by G or by $(T_g)_{g \in G}$, where $T_{g+h} = T_g T_h$.

If $A \subset G$, we will write, for $k \in \mathbb{N}$, kA for the k-fold sum $A + A + \cdots + A$. That is, $kA = \{a_1 + a_2 + \cdots + a_k : a_i \in A,\ 1 \leq i \leq k\}$.

We say that $S \subset G$ is *syndetic* if there is a finite set $F \subset G$ such that $G = S + F$.

Definition. A function $v\colon \mathcal{F} \to G$ is *$\mathcal{F}$-linear* if for every $\alpha, \beta \in \mathcal{F}$ with $\alpha < \beta$, one has $v(\alpha * \beta) = v(\alpha) + v(\beta)$. If $k \in \mathbb{N}$, we shall say such v is *k-covering* if for every $A \subset \mathcal{F}$ with $\overline{d}(A) > 0$, $kv(A) - kv(A)$ is syndetic. We say v is *covering* if it is k-covering for some k.

Elsewhere in the literature, $\mathcal{F}$-linear functions are called *IP systems*. The functions of the following definition, meanwhile, are often called *VIP systems* (of degree at most 2).

Definition. A function $v\colon \mathcal{F} \to G$ is *$\mathcal{F}$-quadratic* if for every $\alpha, \beta, \gamma \in \mathcal{F}$ with $\alpha < \beta < \gamma$, one has

$$v(\alpha * \beta * \gamma) - v(\alpha * \beta) - v(\alpha * \gamma) - v(\beta * \gamma) + v(\alpha) + v(\beta) + v(\gamma) = 0. \tag{1}$$

We remark that $\mathcal{F}$-linear functions are $\mathcal{F}$-quadratic by definition. In practice, we take it that the domain of an $\mathcal{F}$-linear or $\mathcal{F}$-quadratic function v need not be all of $\mathcal{F}$; for example, it is sufficient that v be defined on $\mathcal{F}_n$ for some n.

Proposition 3.1 (cf. [Mc2, Theorem 2.5])**.** *The map $v\colon \mathcal{F} \to G$ is $\mathcal{F}$-linear if and only if there is a sequence $(d_i)_{i\in\mathbb{N}}$ in G such that $v(\alpha) = \sum_{i\in\alpha} d_i$. The map $v\colon \mathcal{F} \to G$ is $\mathcal{F}$-quadratic if and only if there is a matrix $(c_{ij})_{i,j\in\mathbb{N}}$ whose entries lie in G such that $v(\alpha) = \sum_{i,j\in\alpha} c_{ij}$.* □

Note that by replacing c_{ij} by $c_{ij} + c_{ji}$ when $i > j$ we may assume that $c_{ij} = 0$ for $i < j$ in the second part of Proposition 3.1.

According to Proposition 3.1 (with $G = \mathbb{Z}$), Conjecture 1.1 is equivalent to the assertion that $v(\mathcal{F})$ is density intersective for every $\mathcal{F}$-quadratic function v. In the remainder of this section we shall extend the definition of covering to $\mathcal{F}$-quadratic functions and confirm Conjecture 1.1 for all covering $\mathcal{F}$-quadratic functions v.

Proposition 3.2 (cf. [Mc3, Lemma 1.2])**.** *Let $p \in \delta\mathcal{F}$ be idempotent and let $v\colon \mathcal{F} \to G$ be $\mathcal{F}$-quadratic, where G is a commutative Hausdorff topological group. If the limit $g := p\text{-}\lim_\alpha v(\alpha)$ exists, then $g = 0$.*

Proof. Let U be a neighborhood of the identity $0 \in G$ and write $A = \{\gamma : v(\gamma) \in g + U\}$. Then as $p\text{-}\lim_\alpha v(\alpha) = g$, we have $A \in p$. As p is idempotent, one also has $\{\beta : \{\gamma : \beta * \gamma \in A\} \in p\} \in p$. Hence, by requiring also that $\beta, \gamma \in A$,

$$A' := \{\beta : \{\gamma : \beta * \gamma, \beta, \gamma \in A\} \in p\} \in p.$$

Similarly $\{\alpha : \{\beta : \alpha * \beta \in A'\} \in p\} \in p$, and requiring also that $\alpha, \beta \in A'$ gives

$$\{\alpha : \{\beta : \{\gamma : \alpha * \beta * \gamma, \alpha * \beta, \alpha * \gamma, \beta * \gamma, \alpha, \beta, \gamma \in A\} \in p\} \in p\} \in p.$$

Hence there exist $\alpha, \beta, \gamma \in \mathcal{F}$ with $v(\alpha * \beta * \gamma)$, $v(\alpha * \beta)$, $v(\alpha * \gamma)$, $v(\beta * \gamma)$, $v(\alpha)$, $v(\beta)$, and $v(\gamma)$ all lying in $g + U$. Thus by (1), $0 \in 4(g+U) - 3(g+U)$ and so $g \in 3U - 4U$. As U was arbitrary, $g = 0$. □

Definition. Let $\mathcal{H}$ be a separable Hilbert space and G a unitary action on $\mathcal{H}$. Write

$$\mathcal{K}_G = \{f \in \mathcal{H} : \{T_g f : g \in G\} \text{ is precompact in the norm topology}\}.$$

Theorem 3.3 (cf. [Ma]). *$\mathcal{K}_G$ is the closed linear subspace of $\mathcal{H}$ generated by the eigenfunctions of the action (T_g), i.e., by those f for which there is a character $\omega \colon G \to S^1 \subset \mathbb{C}$ such that $T_g f = \omega(g) f$.* □

Let $(X, \mathcal{A}, \mu, G)$ be an invertible measure preserving system on a probability space. For $g \in G$, $x \in X$ and $f \in L^2(X)$, write $T_g f(x) = f(T_g x)$. In this way, G acts unitarily on $L^2(X)$. The action (T_g) is weakly mixing if and only if $\mathcal{K}_G$ is spanned by the constants.

The following theorem is the key to our method; it implies that when p is essential, the weak operator p-limit of $T_{v(\alpha)}$, where v is $\mathcal{F}$-linear and covering, does not depend on v.

Theorem 3.4. *Let $\mathcal{H}$ be a separable Hilbert space and let G be a unitary action on $\mathcal{H}$. Suppose $p \in \delta\mathcal{F}$ is an essential idempotent and let $v \colon \mathcal{F} \to G$ be $\mathcal{F}$-linear and covering. For $f \in \mathcal{H}$ write $Pf = p\text{-}\lim_\alpha T_{v(\alpha)} f$, where the limit is taken in the weak topology. Then P is the orthogonal projection onto $\mathcal{K}_G$.*

Proof. The limit in question exists and satisfies $\|Pf\| \le \|f\|$ because, restricted to closed bounded subsets of $\mathcal{H}$, the weak topology is compact and metrizable. Clearly P is linear, and it is well known that any continuous linear self-map P of a Hilbert space with $\|P\| \le 1$ and $P^2 = P$ is an orthogonal projection. We show now that $P^2 = P$.

Let $f \in \mathcal{H}$ with $\|f\| \le 1$, let $\varepsilon > 0$ and let ρ be a metric for the weak topology on the unit ball of $\mathcal{H}$. Let $A_1 = \{\alpha : \rho(Pf, T_{v(\alpha)} f) < \varepsilon\} \in p$ so that, by idempotence, $\{\alpha : \alpha^{-1} A_1 \in p\} \in p$. Let $A_2 = \{\alpha : \rho(P^2 f, T_{v(\alpha)} Pf) < \varepsilon\} \in p$ and fix $\beta \in A_2 \cap \{\alpha : \alpha^{-1} A_1 \in p\}$. Let

$$A_\beta = \beta^{-1} A_1 \cap \{\gamma > \beta : \rho(P T_{v(\beta)} f, T_{v(\gamma)} T_{v(\beta)} f) < \varepsilon\}.$$

Then $A_\beta \in p$, so in particular A_β is non-empty. Now choose $\gamma \in A_\beta$. One has

$$\rho(P^2 f, Pf) \le \rho(P^2 f, T_{v(\beta)} Pf) + \rho(PT_{v(\beta)} f, T_{v(\gamma)} T_{v(\beta)} f) + \rho(T_{v(\beta*\gamma)} f, Pf) \le 3\varepsilon,$$

where we have used the facts that P commutes with $T_{v(\beta)}$ (an easy exercise), $\beta \in A_2$, $\gamma \in A_\beta$, $\beta * \gamma \in A_1$, and $v(\beta * \gamma) = v(\beta) + v(\gamma)$. Since ε and f were arbitrary, this shows that $P^2 = P$ and hence that P is an orthogonal projection.

Since $\text{range}(P) = \ker(1 - P)$ is a closed linear subspace of $\mathcal{H}$, in order to show that $\mathcal{K}_G \subset \text{range}(P)$ it suffices to show that all eigenfunctions are in $\text{range}(P)$. Suppose we are given an eigenfunction f for (T_g) with eigencharacter $\omega\colon G \to S^1 \subset \mathbb{C}$, so that $T_g f = \omega(g) f$. The limit $p\text{-}\lim_\alpha \omega\big(v(\alpha)\big)$ exists since S^1 is compact. But the function $u\colon \mathcal{F} \to S^1$ defined by $u(\alpha) = \omega\big(v(\alpha)\big)$ is $\mathcal{F}$-linear; that is, one has, for $\alpha < \beta$, $u(\alpha * \beta) = u(\alpha)u(\beta)$. By Proposition 3.2, therefore, $p\text{-}\lim_\alpha \omega\big(v(\alpha)\big) = 1$. From this it easily follows that $p\text{-}\lim_\alpha T_{v(\alpha)} f = f$. Hence $f \in \text{range}(P)$.

Finally we show that $\text{range}(P) \subset \mathcal{K}_G$. Since v is covering, there is k such that v is k-covering. Let $f \in \text{range}(P)$. Then $\|Pf\| = \|T_{v(\alpha)} f\| = \|f\|$, so that $p\text{-}\lim_\alpha T_{v(\alpha)} f$ exists and equals $Pf = f$ in the norm topology (as $p\text{-}\lim_\alpha \|T_{v(\alpha)} f - Pf\|^2 = 2\|Pf\|^2 - 2\,\text{Re}(p\text{-}\lim_\alpha \langle T_{v(\alpha)}, Pf\rangle) = 0$). Let $\varepsilon > 0$, and put $B = \{\alpha : \|T_{v(\alpha)} f - f\| < \varepsilon\}$, so that $B \in p$. Since p is essential, $\overline{d}(B) > 0$, hence $kv(B) - kv(B)$ is syndetic, so we may choose a finite set $R \subset G$ such that every $g \in G$ can be written as $g = r + \sum_{i=1}^k \big(v(b_i) - v(c_i)\big)$, where $b_i, c_i \in B$, $1 \le i \le k$, and $r \in R$. To prove $f \in \mathcal{K}_G$ we need to show that $\{T_g f : g \in G\}$ is precompact, so it is enough to show that $\{T_r f : r \in R\}$ is a $2k\varepsilon$-net for $\{T_g f : g \in G\}$. Let $g \in G$ and write $g = r + \sum_{i=1}^k \big(v(b_i) - v(c_i)\big)$, where $b_i, c_i \in B$, $1 \le i \le k$, and $r \in R$. Then, using the unitarity of T_g,

$$\|T_g f - T_r f\| = \|T_{\sum(v(b_i) - v(c_i))} f - f\| \le \sum_{i=1}^k \|T_{v(b_i)} f - T_{v(c_i)} f\| < 2k\varepsilon.$$

□

We wish to extend the previous theorem to a certain class of $\mathcal{F}$-quadratic functions. This motivates the following definition.

Definition. Let $v\colon \mathcal{F} \to G$ be $\mathcal{F}$-quadratic and let $\alpha \in \mathcal{F}$. The *derivative of v with step α* is given by $D_\alpha v(\beta) = v(\alpha * \beta) - v(\alpha) - v(\beta)$, $\beta > \alpha$. One may easily show that $D_\alpha v$ is $\mathcal{F}$-linear. If $D_\alpha v$ is also covering for all $\alpha \in \mathcal{F}$, we shall say that v is *covering*.

As is typical for proofs of this type, a Van der Corput lemma is used for the extension.

Theorem 3.5 (Van der Corput lemma)**.** *Assume that* $(u_\alpha)_{\alpha\in\mathcal{F}}$ *is a bounded sequence in a Hilbert space. Let* $p \in \delta\mathcal{F}$ *be an idempotent. If* $p\text{-}\lim_\beta p\text{-}\lim_\alpha \langle u_{\beta*\alpha}, u_\alpha\rangle = 0$ *then* $p\text{-}\lim_\alpha u_\alpha = 0$ *in the weak topology.*

Proof. If $\gamma = \{i_1, i_2, \ldots, i_k\}$, where $i_1 < i_2 < \cdots < i_k$, we will write α_γ for $\alpha_{i_1} * \alpha_{i_2} * \cdots * \alpha_{i_k}$. We shall use the convention that $\alpha_\emptyset = \emptyset$.

Without loss of generality we will assume that $\|u_\alpha\| \le 1$, $\alpha \in \mathcal{F}$. Suppose to the contrary that $p\text{-}\lim_\alpha u_\alpha = \tilde{u} \neq 0$. Let $\delta = \frac{\|\tilde{u}\|^2}{2}$ and pick $k \in \mathbb{N}$ and $\varepsilon > 0$ such that $\frac{1}{k} + \varepsilon < \frac{\delta}{2}$. We shall inductively choose an increasing sequence $\alpha_1, \ldots, \alpha_k \in \mathcal{F}$ such that for all j, $1 \le j \le k$, one has

(i) for every non-empty $\gamma, \beta \subset \{1, \ldots, j\}$ with $\beta < \gamma$, $|\langle u_{\alpha_\beta * \alpha_\gamma}, u_{\alpha_\gamma}\rangle| < \varepsilon$;

(ii) for every $\gamma, \beta \subset \{1, \ldots, j\}$ with $\emptyset \neq \beta < \gamma$, $p\text{-}\lim_\alpha |\langle u_{\alpha_\beta * \alpha_\gamma * \alpha}, u_{\alpha_\gamma * \alpha}\rangle| < \varepsilon$;

(iii) for every non-empty $\beta \subset \{1, \ldots, j\}$, $\langle u_{\alpha_\beta}, \tilde{u}\rangle > \delta$;

(iv) for every $\beta \subset \{1, \ldots, j\}$, $\{\omega > \alpha_\beta : \langle u_{\alpha_\beta * \omega}, \tilde{u}\rangle > \delta\} \in p$; and

(v) for every $\beta \subset \{1, \ldots, j\}$, $\{\omega > \alpha_\beta : p\text{-}\lim_\alpha |\langle u_{\alpha_\beta * \omega * \alpha}, u_\alpha\rangle| < \varepsilon\} \in p$.

Having done this, let $v_i = u_{\alpha_1 * \alpha_2 * \cdots * \alpha_i}$, $1 \le i \le k$, and observe that, by (i), $|\langle v_i, v_j\rangle| < \varepsilon$ for all i and j with $1 \le i, j \le k$, $i \neq j$. From this it follows that $\langle \sum_{i=1}^k v_i, \sum_{i=1}^k v_i\rangle < k + k^2\varepsilon < \frac{1}{2}k^2\delta$, which implies that $|\langle \sum_{i=1}^k v_i, \tilde{u}\rangle| \le \|\sum_{i=1}^k v_i\| \, \|\tilde{u}\| < \sqrt{\frac{1}{2}k^2\delta}\sqrt{2\delta} = k\delta$. On the other hand, (iii) implies that $\langle v_i, \tilde{u}\rangle > \delta$ for all i, so that $\langle \sum_{i=1}^k v_i, \tilde{u}\rangle > k\delta$, a contradiction that completes the proof.

Suppose then that $0 \le j < k$ and $\alpha_1, \ldots, \alpha_j$ have been chosen. By the induction hypothesis, for some $\varepsilon' < \varepsilon$,

$$\begin{aligned} B =& \Big(\bigcap_{\beta,\gamma \subset \{1,\ldots,j\},\, \emptyset \neq \beta < \gamma} \{\omega > \alpha_\beta * \alpha_\gamma : |\langle u_{\alpha_\beta * \alpha_\gamma * \omega}, u_{\alpha_\gamma * \omega}\rangle| < \varepsilon'\}\Big) \\ &\cap \Big(\bigcap_{\beta \subset \{1,\ldots,j\}} \{\omega > \alpha_\beta : \langle u_{\alpha_\beta * \omega}, \tilde{u}\rangle > \delta\}\Big) \\ &\cap \Big(\bigcap_{\beta \subset \{1,\ldots,j\}} \{\omega > \alpha_\beta : p\text{-}\lim_\alpha |\langle u_{\alpha_\beta * \omega * \alpha}, u_\alpha\rangle| < \varepsilon\}\Big) \\ =& B_1 \cap B_2 \cap B_3 \end{aligned}$$

is a member of p. (Briefly, $B_1 \in p$ by (ii), $B_2 \in p$ by (iv) and $B_3 \in p$ by (v).) As p is idempotent, we may choose $\alpha_{j+1} \in B$ such that $\alpha_{j+1}^{-1} B \in p$.

One now checks that (i)–(v) hold for j replaced by $j+1$. A few details: (i) follows from $\alpha_{j+1} \in B_1$, (ii) follows from $\alpha_{j+1} \in B_3$ if $j+1 \in \beta$ and from $\alpha_{j+1}^{-1} B_1 \in p$ if $j+1 \in \gamma$, (iii) follows from $\alpha_{j+1} \in B_2$, (iv) follows from $\alpha_{j+1}^{-1} B_2 \in p$ and (v) follows from $\alpha_{j+1}^{-1} B_3 \in p$. □

Here is the extension to covering $\mathcal{F}$-quadratic functions.

Theorem 3.6. *Let $\mathcal{H}$ be a separable Hilbert space and let (T_g) be a unitary G-action on $\mathcal{H}$. Let $p \in \delta\mathcal{F}$ be an essential idempotent and suppose $v\colon \mathcal{F} \to G$ is $\mathcal{F}$-quadratic and covering. For $f \in \mathcal{H}$, write $Pf = p\text{-}\lim_\alpha T_{v(\alpha)} f$, where the limit is taken in the weak topology. Then P is the orthogonal projection onto $\mathcal{K}_G$.*

Proof. As in the proof of Theorem 3.4, we must show that $P = P^2$. Let $f \in \mathcal{H}$ and write $f = f_1 + f_2$, where $f_1 \in \mathcal{K}_G$ and $f_2 \in \mathcal{K}_G^\perp$. For $\beta \in \mathcal{F}$ and $h \in \mathcal{H}$, write $P_\beta h = p\text{-}\lim_\alpha T_{D_\beta v(\alpha)} h$. Since $D_\beta v$ is $\mathcal{F}$-linear and covering, by Theorem 3.4 P_β is the orthogonal projection onto $\mathcal{K}_G$. Hence, writing $x_\alpha = T_{v(\alpha)} f_2$,

$$\begin{aligned} p\text{-}\lim_\beta p\text{-}\lim_\alpha \langle x_\alpha, x_{\beta * \alpha}\rangle &= p\text{-}\lim_\beta p\text{-}\lim_\alpha \langle f_2, T_{v(\beta*\alpha)-v(\alpha)} f_2\rangle \\ &= p\text{-}\lim_\beta p\text{-}\lim_\alpha \langle T_{-v(\beta)} f_2, T_{D_\beta v(\alpha)} f_2\rangle \\ &= p\text{-}\lim_\beta \langle T_{-v(\beta)} f_2, P_\beta f_2\rangle = 0. \end{aligned}$$

By Theorem 3.5, one has $p\text{-}\lim_\alpha x_\alpha = 0$ weakly; that is, $Pf_2 = 0$. On the other hand, just as in the proof of Theorem 3.4, one has $Pf_1 = f_1$, by Proposition 3.2. (Note for this step that the map $\alpha \to \omega\big(v(\alpha)\big)$ is $\mathcal{F}$-quadratic.) □

Now by a standard argument, a projection theorem yields a recurrence theorem.

Corollary 3.7. *Let $(X, \mathcal{A}, \mu, G)$ be a measure preserving system, $p \in \delta\mathcal{F}$ an essential idempotent and $v\colon \mathcal{F} \to G$ $\mathcal{F}$-quadratic and covering, and suppose $\mu(A) > 0$. Then $p\text{-}\lim_\alpha \mu(A \cap T_{v(\alpha)} A) \geq \mu(A)^2$.*

Proof. Let $\mathcal{H} = L^2(X)$ and $f = 1_A \in L^2(X)$. Then one has $p\text{-}\lim_\alpha \mu(A \cap T_{v(\alpha)} A) = p\text{-}\lim_\alpha \langle f, T_{-v(\alpha)} f\rangle = \langle f, Pf\rangle = \langle Pf, Pf\rangle \geq \mu(A)^2$. (For the final inequality, we used the fact that P is the orthogonal projection onto a space containing the constants.) □

Combined with the Furstenberg correspondence principle, Corollary 3.7 is sufficient to achieve the primary goal of this section, namely showing that $v(\mathcal{F})$ is density intersective for any covering $\mathcal{F}$-quadratic function v. It remains to give interesting examples of covering $\mathcal{F}$-quadratic functions.

4 Examples of covering

In this section we obtain specific applications of Corollary 3.7 as well as additional examples of covering. First we give some background material. See, e.g., [BHiM] for more details.

Any countable discrete abelian group G admits of a *Følner sequence*, i.e., an exhaustive sequence (Φ_n) of finite subsets of G satisfying, for every $g \in G$, $\frac{|\Phi_n \cap (g+\Phi_n)|}{|\Phi_n|} \to 1$ as $n \to \infty$. Any Følner sequence, in turn, gives rise to a notion of upper density: $\overline{d}_\Phi(A) = \limsup_n \frac{|A \cap \Phi_n|}{|\Phi_n|}$. Such densities are shift invariant: $\overline{d}_\Phi(g + A) = \overline{d}_\Phi(A)$ for $A \subset G$ and $g \in G$.

Although G may contain countably many disjoint sets of upper density 1, this is not so for shifts of the same set. Indeed, if $\overline{d}_\Phi(A) > \frac{1}{k}$ then G cannot contain k disjoint shifts of A. It follows that if $\overline{d}_\Phi(A) > 0$ then $A - A$ meets every *difference set* $D = \{g_i - g_j : i > j\}$. (Here $(g_i)_{i\in\mathbb{N}}$ is any infinite sequence of elements of G.) A *thick* set is a subset of G that meets every syndetic set (conversely, a set is syndetic if and only if it meets every thick set). Alternatively, $T \subset G$ is thick if for every finite set F, there is some $g \in G$ such that $g + F \subset T$. It is easy to show that any thick set contains a difference set. Therefore if $\overline{d}_\Phi(A) > 0$ then any thick set meets $A - A$. In other words, $A - A$ is syndetic. This leads to the following.

Lemma 4.1. *Let (Φ_n) be a Følner sequence for G and let $v\colon \mathcal{F} \to G$ be $\mathcal{F}$-linear. Suppose that for some $k \in \mathbb{N}$ and every $A \subset \mathcal{F}$ with $\overline{d}(A) > 0$, one has $\overline{d}_\Phi\big(kv(A)\big) > 0$. Then v is k-covering.*

Proof. Immediate as $\overline{d}_\Phi\big(kv(A)\big) > 0$ implies $kv(A) - kv(A)$ is syndetic. □

Some of our examples require the following theorem.

Theorem 4.2 (cf. [BKMP, Corollary 1])**.** *Assume $k, j, n \in \mathbb{N}$ with $j \le n$ and let B be a subset of $\{0,1\}^n$, which we view as a subset of $\bigoplus_{i\in\mathbb{N}} \mathbb{Z}_{k+1}$. Suppose moreover that $|B| \ge 2^j$. Then $|kB| \ge (k+1)^j$.* □

We shall use Theorem 4.2 via the following theorem concerning $\bigoplus_{i\in\mathbb{N}}\mathbb{Z}_{k+1}$. It will assist in the proof of Lemma 4.5 below.

Theorem 4.3. *Suppose $k\in\mathbb{N}$ and define $G=\bigoplus_{i\in\mathbb{N}}\mathbb{Z}_{k+1}$, with $(e_i)_{i\in\mathbb{N}}$ its standard generating set ($e_1=(1,0,0,\dots)$, $e_2=(0,1,0,\dots)$, etc.). For $\alpha\in\mathcal{F}$, define $v(\alpha)=\sum_{i\in\alpha}e_i$. Then v is $\mathcal{F}$-linear and k-covering.*

Proof. Let $A\subset\mathcal{F}$ with $\overline{d}(A)>0$. Choose t large enough that $\overline{d}(A)>2^{-t}$. Let $\Phi_n=\{a_1e_1+\cdots+a_ne_n : a_i\in\mathbb{Z}_{k+1},\ 1\le i\le n\}$. Then $(\Phi_n)_{n\in\mathbb{N}}$ is a Følner sequence. We will show that $\overline{d}_\Phi\big(kv(A)\big)>(k+1)^{-t}$, which will be sufficient for the proof by Lemma 4.1. Let n_0 be arbitrary and choose $n>n_0$ such that $|A\cap\mathcal{F}\big(\{1,2,\dots,n\}\big)|\ge 2^{n-t}$. Setting $A'=A\cap\mathcal{F}\big(\{1,2,\dots,n\}\big)$ we may apply Theorem 4.2 and conclude that $|kv(A')|\ge(k+1)^{n-t}$. Since $kv(A')\subset\Phi_n$, this yields $\frac{|kv(A)\cap\Phi_n|}{|\Phi_n|}\ge\frac{|kv(A')\cap\Phi_n|}{|\Phi_n|}\ge(k+1)^{-t}$. Since n_0 was arbitrary and $n>n_0$, we are done. □

We shall not make use of the following optional corollary concerning weak mixing actions of $\bigoplus_{i\in\mathbb{N}}\mathbb{Z}_{k+1}$, however it demonstrates nicely what is going on in the results for $\mathbb{Z}$ to come. We include it for aficionados, who may be intrigued to see the conclusion following without the stronger hypothesis of *mild mixing*.

Corollary 4.4. *Assume G and v are as in Theorem 4.3. Let $(X,\mathcal{A},\mu,G)$ be a weakly mixing measure preserving probability system and let $p\in\delta\mathcal{F}$ be an essential idempotent. Then for any $f,g\in L^2(X)$, one has*

$$p\text{-}\lim_\alpha\int fT_{v(\alpha)}g\,d\mu=\Big(\int f\,d\mu\Big)\Big(\int g\,d\mu\Big).$$

Proof. Since G is weakly mixing, $\mathcal{K}_G$ consists of the constant functions. Hence by Theorem 3.4, $p\text{-}\lim_\alpha T_{v(\alpha)}g=Pg$ in the weak topology and $Pg=\int g\,d\mu$ is the projection onto $\mathcal{K}_G$. Thus

$$p\text{-}\lim_\alpha\int fT_{v(\alpha)}g=\int f\big(Pg\big)\,d\mu=\int f\Big(\int g\,d\mu\Big)\,d\mu=\Big(\int f\,d\mu\Big)\Big(\int g\,d\mu\Big).$$

□

Lemma 4.5. *Fix $k\in\mathbb{N}$ and let $(d_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers. Suppose there exists $M>0$ such that $k\big(\sum_{i=1}^n d_i\big)<d_{n+1}<M(k+1)^{n+1}$ for every large enough $n\in\mathbb{N}$. If $u\colon\mathcal{F}\to\mathbb{Z}$ is defined by $u(\alpha)=\sum_{i\in\alpha}d_i$, then u is $\mathcal{F}$-linear and k-covering.*

Proof. We assume in the proof that the given string of inequalities holds for all n; the reader may make the minor adjustments for the general case. Let $\Phi_n = \{1, 2, \ldots, M(k+1)^{n+1}\}$, $n \in \mathbb{N}$. Then (Φ_n) is a Følner sequence. Let $A \in \mathcal{F}$ with $\overline{d}(A) > 0$. Choose t large enough that $\overline{d}(A) > 2^{-t}$. Let n_0 be arbitrary and pick $n > n_0$ having the property that $A' = A \cap \mathcal{F}\big(\{1, 2, \ldots, n\}\big)$ satisfies $|A'| > 2^{n-t}$. Letting $v \colon \mathcal{F}\big(\{1, 2, \ldots, n\}\big) \to \mathbb{Z}_{k+1}^n$ be as in the proof of Theorem 4.3, one has, as was the case in that proof, $|kv(A')| \geq (k+1)^{n-t}$. Now define $\pi \colon \mathbb{Z}_{k+1}^n \to \Phi_n$ by $\pi(a_1, a_2, \ldots, a_n) = a_1 d_1 + a_2 d_2 + \cdots + a_n d_n$. The restrictions on (d_i) entail that π is one-to-one, hence $\big|\pi\big(kv(A')\big)\big| \geq (k+1)^{n-t}$. But by linearity of π, $\pi\big(kv(A')\big) = k\pi\big(v(A')\big)$. Moreover, it is easily checked that $\pi\big(v(A')\big) = u(A')$. Therefore $|ku(A')| \geq (k+1)^{n-t}$. But $ku(A') \subset \Phi_n$, so $\frac{|ku(A) \cap \Phi_n|}{|\Phi_n|} \geq \frac{(k+1)^{-t-1}}{M}$. Since n_0 was arbitrary and $n > n_0$, we have established that $\overline{d}_\Phi\big(ku(A)\big) \geq \frac{(k+1)^{-t-1}}{M}$, and Lemma 4.1 applies. □

Combining Lemma 4.5 with Corollary 3.7, we already get new results.

Corollary 4.6. *Fix $k \in \mathbb{N}$ and assume that $(c_{ij})_{i,j \in \mathbb{N}}$ is an integer matrix such that $c_{ij} = 0$ when $j > i$ and each column $(d_i = c_{ij})$ satisfies the rate-of-growth condition of Lemma 4.5. Write $v(\alpha) = \sum_{i,j \in \alpha} c_{ij}$, $\alpha \in \mathcal{F}$. Then if $(X, \mathcal{A}, \mu, T)$ is an invertible measure preserving probability system, $\mu(A) > 0$, and $p \in \delta\mathcal{F}$ is an essential idempotent, then $p\text{-}\lim_\alpha \mu(A \cap T_{v(\alpha)} A) \geq \mu(A)^2$.*

Proof. One has $D_\alpha v(\beta) = \sum_{i \in \beta} d_i$, where $d_i = \sum_{j \in \alpha} c_{ij}$. Since each column of the matrix satisfies the rate-of-growth condition of Lemma 4.5, so does (d_i), which is a finite sum of columns. By Lemma 4.5, $D_\alpha v$ is k-covering, which implies, as α is arbitrary, that v is covering. Hence the conclusion follows from Corollary 3.7. □

Corollary 4.7 (of the proof of Lemma 4.5)**.** *Fix $k \in \mathbb{N}$ and let $(d_n)_{n \in \mathbb{N}}$ be a sequence of natural numbers. Suppose there exists $M > 0$ and a one-to-one sequence (m_i) in $\mathbb{N}$ such that $k\big(\sum_{i=1}^n d_{m_i}\big) < d_{m_{n+1}} < M(k+1)^{n+1}$ for every large enough $n \in \mathbb{N}$. If $u \colon \mathcal{F} \to \mathbb{Z}$ is defined by $u(\alpha) = \sum_{i \in \alpha} d_i$, then u is $\mathcal{F}$-linear and k-covering.*

Proof. We use the fact that if Φ_n is a Følner sequence and (x_n) is an arbitrary sequence then $\Psi_n = \Phi_n + x_n$ defines a Følner sequence (Ψ_n).

Modify the proof of Lemma 4.5 as follows. Once n is chosen, pick $N > m_n$ such that $A' = A \cap \mathcal{F}\big(\{1, 2, \ldots, N\}\big)$ satisfies $|A'| > 2^{N-t}$. For $\alpha \subset \{1, 2, \ldots, N\} \setminus \{m_1, \ldots, m_n\}$ write $A_\alpha = \{B \subset \{m_1, \ldots, m_n\} : \alpha \cup B \in A'\}$. As $\sum_\alpha |A_\alpha| = |A'|$, we can choose $\alpha = \alpha_n$ so that $|A_{\alpha_n}| > 2^{n-t}$. Run the rest of the proof with A_{α_n} in place of A' to get $\frac{|ku(A_{\alpha_n}) \cap \Phi_n|}{|\Phi_n|} \geq \frac{(k+1)^{-t-1}}{M}$, which implies that $\frac{|ku(A) \cap (x_n + \Phi_n)|}{|x_n + \Phi_n|} \geq \frac{(k+1)^{-t-1}}{M}$, where

$x_n = ku(\alpha_n)$. One concludes that $\overline{d}_\Psi\big(ku(A)\big) \geq \frac{(k+1)^{-t-1}}{M}$, where $\Psi_n = x_n + \Phi_n$. In particular, $ku(A) - ku(A)$ is syndetic. □

Corollary 4.8. *Let (d_n) be an unbounded sequence of natural numbers and set $v(\alpha) = \sum_{n\in\alpha} d_n$. Define $r_n = \min_{1\leq y<n} \frac{d_n}{d_y}$. Suppose there is a $k \in \mathbb{N}$ such that for every sequence of indices (m_n) with $d_{m_n} > k^n$ one has $\sum_{n=1}^\infty (r_{m_n} - 1) < \infty$. Then v is covering.*

Proof. Let m_1 be the least integer such that $d_{m_1} > k$. Having chosen $m_1, \ldots, m_n$, let m_{n+1} be the least index satisfying $k\sum_{i=1}^n d_{m_i} < d_{m_{n+1}}$. The sequence $(m_n)_{n\in\mathbb{N}}$ is increasing, so one-to-one, and $d_{m_{n+1}} > kd_{m_n}$, so $d_{m_n} > k^n$ by induction on n. By Corollary 4.7 we need only find M such that $d_{m_{n+1}} < M(k+1)^{n+1}$ for all n. Put $N_n = k\sum_{i=1}^n d_{m_i}$. Since $k\sum_{i=1}^n d_{m_i} \geq d_y$ for $y < m_{n+1}$, one has $d_{m_{n+1}} \leq r_{m_{n+1}} N_n$. Therefore, $N_{n+1} = N_n + kd_{m_{n+1}} \leq (1 + kr_{m_{n+1}})N_n$. Since $\sum_{n=1}^\infty (r_{m_n} - 1) < \infty$, $r_{m_{n+1}}$ is bounded and the product $\prod_n \left(\frac{1+kr_{m_n}}{1+k}\right) = \prod_n \left(1 + \frac{k(r_{m_n}-1)}{1+k}\right)$ converges. Hence $d_{m_{n+1}} < r_{m_{n+1}} N_1 \prod_{i=2}^{n+1}(1 + kr_{m_i}) \leq M(k+1)^{n+1}$ for some M independent of n. □

Examples. The map $v(\alpha) = \sum_{n\in\alpha} d_n$ is covering by Corollary 4.8 for a great many sequences (d_n), including the following:

1. $d_n = \lfloor n^\gamma \rfloor$, where $\gamma > 0$.
2. $d_n = \lfloor \exp(n^\gamma) \rfloor$, where $0 < \gamma < \frac{1}{2}$.

We sketch a justification of 2. In this case $r_x \approx \exp\big(x^\gamma - (x-1)^\gamma\big) \approx \exp\big(\gamma x^{\gamma-1}\big)$, so $r_{x^2} - 1 \approx \exp\big(\gamma x^{2\gamma-2}\big) - 1 \approx \gamma x^{2\gamma-2}$. Also, if $d_{m_x} > 3^x$ then $m_x^\gamma > x$ and so $m_x > x^2$. Thus

$$\sum_{x=1}^\infty (r_{m_x} - 1) < \sum_{x=1}^\infty (r_{x^2} - 1) \approx \sum_{x=1}^\infty \gamma x^{2\gamma-2} < \infty.$$

The following example shows what can go wrong when one has no control on the sequence (r_n) defined in the proof of Corollary 4.8.

Proposition 4.9. *Let $(s_n)_{n=1}^\infty$ be any sequence of natural numbers converging to ∞. Then there exists a sequence $(d_n)_{n=1}^\infty$ such that $1 \leq d_n \leq s_n\sqrt{n}$ for all n and $v(\alpha) = \sum_{n\in\alpha} d_n$ is not covering.*

Proof. Let (m_i) be a rapidly increasing sequence of natural numbers. Set $d_n = 1$ for all n with $1 \le n \le \frac{m_1^2}{2}$, and

$$d_n = m_1 m_2 \dots m_t, \quad \text{for} \quad \tfrac{m_1^2}{2} + \tfrac{m_2^2}{4} + \dots \tfrac{m_t^2}{2^t} < n \le \tfrac{m_1^2}{2} + \tfrac{m_2^2}{4} + \dots + \tfrac{m_t^2}{2^t} + \tfrac{m_{t+1}^2}{2^{t+1}}.$$

One may check that if (m_i) grows rapidly enough then $1 \le d_n \le s_n \sqrt{n}$ holds for all n.

Now define a set $A \subset \mathcal{F}$ as follows. For $t \in \mathbb{N}$, let

$$A_t = \Big\{ B \subset \mathcal{F}\Big(\{ \sum_{i=1}^{t-1} \tfrac{m_i^2}{2^i} + 1, \dots, \sum_{i=1}^{t} \tfrac{m_i^2}{2^i} \} \Big) : \tfrac{m_t^2}{2^{t+1}} - \tfrac{tm_t}{2^{t/2+1}} < |B| < \tfrac{m_t^2}{2^{t+1}} + \tfrac{tm_t}{2^{t/2+1}} \Big\}.$$

(Roughly, B consists of those subsets of $\{\sum_{i=1}^{t-1} \frac{m_i^2}{2^i} + 1, \dots, \sum_{i=1}^{t} \frac{m_i^2}{2^i}\}$ having cardinality within t standard deviations of expected were B chosen by coin tossing.)

The relative density z_t of A_t in $\mathcal{F}\big(\{\sum_{i=1}^{t-1} \frac{m_i^2}{2^i} + 1, \dots, \sum_{i=1}^{t} \frac{m_i^2}{2^i}\}\big)$ increases to 1 fast enough (e.g., by the central limit theorem) to ensure that $\prod_{t=1}^{\infty} z_t > 0$. From this, we get that

$$A = \big\{ \alpha_1 \cup \alpha_2 \cup \dots \cup \alpha_n : n \in \mathbb{N},\, \alpha_t \in A_t,\, 1 \le t \le n \big\}$$

satisfies $\overline{d}(A) > 0$.

Put $v(\alpha) = \sum_{n \in \alpha} d_n$ and let $k \in \mathbb{N}$. We claim that $kv(A) - kv(A)$ is not syndetic; indeed does not have positive density. To see this, note that $\alpha \in A$ can be written $\alpha = \alpha_1 \cup \alpha_2 \cup \dots \cup \alpha_N$, where $\alpha_t \in A_t$, $1 \le t \le N$. We then have

$$v(\alpha) = |\alpha_1| + m_1|\alpha_2| + m_1 m_2 |\alpha_3| + \dots + m_1 m_2 \dots m_{N-1} |\alpha_N|,$$

with $|\alpha_t|$ confined to an interval of length $tm_t 2^{-t/2}$. It follows that for any $x \in kv(A) - kv(A)$

$$x \equiv x_1 + m_1 x_2 + m_1 m_2 x_3 + \dots + m_1 m_2 \dots m_{n-1} x_n \mod m_1 m_2 \dots m_n,$$

where x_t is confined to an interval of length $2ktm_t 2^{-t/2}$. It follows that, modulo $m_1 m_2 \dots m_n$, $kv(A) - kv(A)$ hits at most $\prod_{t=1}^{n} 2ktm_t 2^{-t/2}$ residue classes, and hence has density at most $(2k)^n n! 2^{-n(n+1)/4} \to 0$ as $n \to \infty$. □

On the other hand, for somewhat slower growing sequences (d_n), one may prove a positive result, irrespective of control on the sequence (r_n) defined in Corollary 4.8. We begin with the following concentration function estimate.

Lemma 4.10. *There exist positive constants c, C, having the following properties. Suppose $N \in \mathbb{N}$ and $(d_n)_{n=1}^N$ is a sequence of integers with $d_1 = 1$ and $1 \le d_n \le \max\{1, c\sqrt{\frac{n}{\log n}}\}$ for $n \ge 2$. If $(X_n)_{n=1}^N$ are independent random variables with $\mathbb{P}(X_i = 0) = \frac{1}{2} = \mathbb{P}(X_i = d_i)$ then $\mathbb{P}(\sum_{n=1}^N X_n = k) \le C(\sum_{n=1}^N d_n^2)^{-1/2}$ for all k.*

Proof. Let $c = \frac{1}{11}$ and $C = 3$; we have made very little attempt to make these constants optimal. We may assume without loss of generality that $(d_n)_{n=1}^N$ is non-decreasing. If $d_1 = d_2 = \cdots = d_N = 1$ then $\mathbb{P}(\sum_{n=1}^N X_n = k) \le \binom{n}{\lfloor \frac{n}{2} \rfloor} 2^{-n} \le \frac{C}{\sqrt{N}}$ for all k; we may therefore assume that $d_N \ge 2$. In particular, this implies that $d_N \le c\sqrt{\frac{N}{\log N}}$.

Write $X = \sum_{n=1}^N X_n$. Then for $\omega \in \mathbb{R}$ and $k \in \mathbb{Z}$,

$$\mathbb{E}(e^{2\pi i\omega(X-k)}) = e^{-2\pi i\omega k} \prod_{n=1}^N \mathbb{E}(e^{2\pi i\omega X_n}) = e^{-2\pi i\omega k} \prod_{n=1}^N \cos(\pi d_n \omega) e^{\pi i\omega d_n}.$$

Integrating with respect to ω,

$$\mathbb{E}(1_{X=k}) = \mathbb{E}\Big(\int_0^1 e^{2\pi i\omega(X-k)}\, d\omega \Big) = \int_0^1 \mathbb{E}(e^{2\pi i\omega(X-k)})\, d\omega.$$

It follows that

$$\mathbb{P}(X = k) \le \int_0^1 \big|\mathbb{E}(e^{2\pi i\omega(X-k)})\big|\, d\omega = \int_0^1 \prod_{n=1}^N |\cos(\pi d_n \omega)|\, d\omega.$$

One has $|\cos x| \le e^{-x^2/2}$ for $|x| \le .56\pi$. If $1 \le n \le N$, choose $t_n \in \mathbb{Z}$ and $\{d_n\omega\} \in [-.56, .56]$ such that $d_n\omega = t_n + \{d_n\omega\}$. (This representation may not be unique, which will be important later.) Then

$$|\cos(\pi d_n \omega)| = |\cos \pi\{d_n\omega\}| \le e^{-\frac{\pi^2}{2}\{d_n\omega\}^2} = e^{-\frac{\pi^2}{2}(d_n\omega - t_n)^2}.$$

Thus

$$\mathbb{P}(X = k) \le \int_0^1 \exp\Big(-\frac{\pi^2}{2} \sum_{n=1}^N (d_n\omega - t_n)^2 \Big)\, d\omega. \tag{2}$$

Write $V = \sum_{n=1}^N d_n^2$, and more generally $V_S = \sum_{n\in S} d_n^2$ when $S \subset \{1, \ldots, N\}$. Since $d_N \le c\sqrt{\frac{N}{\log N}}$,

$$V^{1/2} = \Big(\sum_{n=1}^N d_n^2 \Big)^{1/2} \le (N d_N^2)^{1/2} \le c\frac{N}{(\log N)^{1/2}} \le N. \tag{3}$$

According to (2), it suffices to show that

$$V^{1/2}\int_0^1 \exp\Big(-\tfrac{\pi^2}{2}\sum_{n=1}^{N}(d_n\omega - t_n)^2\Big)\,d\omega \le C = 3. \tag{4}$$

By (3), the contribution in the left hand side of (4) from those ω for which there exists a choice (t_n) making the integrand less than $\frac{1}{N}$ (i.e., for which there are t_n with $\frac{\pi^2}{2}\sum_{n=1}^{N}(d_n\omega - t_n)^2 > \log N$) is at most $N \cdot \frac{1}{N} = 1$.

For a fixed choice (t_n), the function $g(\omega) = \frac{\pi^2}{2}\sum_{n=1}^{N}(d_n\omega - t_n)^2$ is quadratic in ω and can be written in the form $g(\omega) = A(\omega-\omega_0)^2 + B$, where $A = \frac{\pi^2}{2}\sum_{n=1}^{N} d_n^2 = \frac{\pi^2}{2}V$ and B is the minimum value of g. It follows that the contribution to the left hand side of (4) from ω giving rise to this choice of (t_n) is at most

$$V^{1/2}\int_{-\infty}^{\infty}\exp\big(-A(\omega-\omega_0)^2 - B\big)\,d\omega = V^{1/2}\sqrt{\tfrac{\pi}{A}}\,e^{-B} = (\pi/2)^{-1/2}e^{-B}.$$

Therefore, it suffices to show that $\sum_{(t_n)}(\pi/2)^{-1/2}e^{-B} < 2$, where the sum is over those choices (those we choose to make in the remainder of the proof) of the sequence (t_n) for which $B \le \log N$. We will in fact show that $\sum_{(t_n)} e^{-B} < 2.1$.

Setting $g'(\omega) = 0$ and solving for ω, we get $\omega_0\sum_{n=1}^{N} d_n^2 = \sum_{n=1}^{N} d_n t_n$. Thus

$$\begin{aligned}
B = g(\omega_0) &= \frac{\pi^2}{2}\sum_n\Big(d_n\frac{\sum d_j t_j}{\sum d_j^2} - t_n\Big)^2\\
&= \frac{\pi^2}{2}\Big(\sum d_n^2\frac{(\sum d_j t_j)^2}{(\sum d_j^2)^2} - 2\sum d_n t_n\frac{\sum d_j t_j}{\sum d_j^2} + \sum t_n^2\Big)\\
&= \frac{\pi^2}{2}\Big(\frac{(\sum d_j t_j)^2}{\sum d_j^2} - 2\frac{(\sum d_n t_n)^2}{\sum d_j^2} + \frac{\sum t_n^2\sum d_j^2}{\sum d_j^2}\Big)\\
&= \frac{\pi^2}{2}V^{-1}\Big((\sum t_n^2)(\sum d_n^2) - (\sum d_n t_n)^2\Big)\\
&= \frac{\pi^2}{2}V^{-1}\Big(\sum_{i<j}(d_i^2t_j^2 + d_j^2t_i^2) + \sum d_n^2t_n^2 - \sum_{i,j} d_it_id_jt_j\Big)\\
&= \frac{\pi^2}{2}V^{-1}\sum_{i<j}(d_i^2t_j^2 + d_j^2t_i^2 - 2d_id_jt_it_j)\\
&= \frac{\pi^2}{2}V^{-1}\sum_{i<j}(d_it_j - d_jt_i)^2.
\end{aligned} \tag{5}$$

We now discuss the choice of (t_n). Recall, we only need consider those ω for which *all* legal choices (t_n) give $g(\omega) \leq \log N$. For such ω, initially we will choose t_n such that $\{d_n\omega\} \in [-.5, .5]$ (we will change some of the t_n in a moment). Define an equivalence relation $\sim$ on $\{1, 2, \ldots, N\}$ by $i \sim j$ if and only if $\frac{t_i}{d_i} = \frac{t_j}{d_j}$. Let S be a largest equivalence class of $\sim$, and choose relatively prime a, d, with $\frac{a}{d}$ equal to the common value $\frac{t_i}{d_i}$, $i \in S$. Note in particular that $d \mid d_i$ for all $i \in S$. Since each i has at least $N - |S|$ values of j for which $i \not\sim j$,

$$\sum_{i<j} (d_i t_j - d_j t_i)^2 \geq \tfrac{1}{2} N(N - |S|).$$

But $B \leq g(\omega) \leq \log N$, so by (5) and (3) one has

$$\sum_{i<j} (d_i t_j - d_j t_i)^2 \leq \frac{2}{\pi^2} V \log N \leq \frac{2c^2}{\pi^2} N^2. \tag{6}$$

Since $c < 1$ we deduce that

$$\tfrac{1}{2} N(N - |S|) \leq \tfrac{2c^2}{\pi^2} N^2 < \tfrac{1}{4} N^2,$$

and so $|S| > \frac{1}{2}N$. Thus the largest equivalence class S is in fact unique. We shall now strengthen this bound on S by showing that $S^c := \{1, \ldots, N\} \setminus S$ is rather small.

If $i \in S$ then $(d_i t_j - d_j t_i)^2$ is divisible by $(\frac{d_i}{d})^2$, so by considering pairs i, j, exactly one of which is in S, one gets

$$\sum_{i<j} (d_i t_j - d_j t_i)^2 \geq |S^c| \frac{1}{d^2} V_S. \tag{7}$$

Combining this with (6), $V_{S^c} \leq |S^c| d_N^2$, and $V_S \geq |S| d^2 \geq \frac{1}{2} d^2 N$ we obtain

$$\begin{aligned} \frac{\pi^2}{2} |S^c| \frac{1}{d^2} V_S &\leq V \log N \\ &\leq V_S \log N + |S^c| d_N^2 \log N \\ &\leq V_S \log N + c^2 |S^c| N \\ &= V_S \log N + 2c^2 d^{-2} |S^c| (\tfrac{1}{2} d^2 N) \\ &\leq V_S \log N + 2c^2 d^{-2} |S^c| V_S. \end{aligned}$$

Canceling V_S and isolating $|S^c|$, we get

$$|S^c| \leq \frac{d^2 \log N}{\frac{\pi^2}{2} - 2c^2} < \frac{1}{4} d^2 \log N. \tag{8}$$

Our next task is to estimate the closeness of ω to $\frac{a}{d}$. We have

$$\frac{1}{2}N(\omega - \tfrac{a}{d})^2 \le (\omega - \tfrac{a}{d})^2 \sum_{n\in S} d_n^2 = \sum_{n\in S} (\omega - \tfrac{t_n}{d_n})^2 d_n^2$$
$$\le \sum_{n=1}^{N} (d_n\omega - t_n)^2 = \frac{2}{\pi^2} g(\omega) \le \frac{2}{\pi^2}\log N \le \frac{2c^2}{\pi^2 d_N^2} N.$$

Thus $|\omega - \frac{a}{d}| \le \frac{2c}{\pi d_N}$, and so $|d_n\omega - d_n\frac{a}{d}| \le \frac{2c}{\pi} < .06$ for all $n \in S^c$. What this means is that if we rechoose (for all $n \in S^c$) t_n such that $d_n\frac{a}{d} - t_n \in (-.5, .5]$, the sequence (t_n) will still be legal for ω, i.e., $|d_n\omega - t_n| \le .56$. By choosing in this fashion, we ensure that for each fixed d, at most $d+1$ sequences (t_n) contribute, there being $d+1$ choices for a, while a and d determine (t_n) uniquely.

Now by (8), $|S^c| \le \frac{1}{4}d^2 \log N$, so $V_{S^c} \le |S^c| d_N^2 \le \frac{1}{4}c^2 d^2 N$. But $V_S \ge \frac{1}{2}d^2 N$, so $V_S \ge \frac{1}{2}V$. Therefore, using (7),

$$\sum_{i<j} (d_i t_j - d_j t_i)^2 \ge |S^c| \frac{1}{d^2} V_S \ge |S^c| \frac{1}{2d^2} V. \tag{9}$$

Let $n \in S$. If $d > 1$ then $d \le d_n \le c\sqrt{\frac{n}{\log n}}$, which implies that $n \ge \frac{d^2 \log n}{c^2} > \frac{d^2 \log d}{c^2} + 1$. It follows that all integers $1, \ldots, \lceil \frac{d^2 \log d}{c^2} \rceil$ lie in S^c, and so $|S^c| \ge \frac{d^2 \log d}{c^2}$. As this obviously holds for $d = 1$ as well, (5) and (9) imply that

$$B = \frac{\pi^2}{2} V^{-1} \sum_{i<j} (d_i t_j - d_j t_i)^2 \ge \frac{\pi^2}{4d^2} |S^c| \ge \frac{\pi^2}{4c^2} \log d \ge 100 \log d.$$

Thus

$$\sum_{(t_n)} e^{-B} \le \sum_{d=1}^{\infty} (d+1) e^{-100 \log d} = \sum_{d=1}^{\infty} \frac{d+1}{d^{100}} < 2.1$$

as required. $\square$

Theorem 4.11. *There exists an absolute constant $c > 0$ such that if $1 \le d_n \le c\sqrt{\frac{n}{\log n}}$ for all large enough n then for every $A \in \mathcal{F}$ with $\overline{d}(A) > 0$, $d^*\big(v(A)\big) > 0$, where $v(\alpha) = \sum_{n\in\alpha} d_n$. In particular v is 1-covering.*

Proof. We may assume without loss of generality that the inequality in question holds for all n, since the behavior of $(d_n)_{n=1}^{N}$ can affect $d^*\big(v(A)\big)$ by at most a factor of 2^{-N}.

Let $(X_n)_{n=1}^\infty$ be independent random variables with $\mathbb{P}(X_i = 0) = \frac{1}{2} = \mathbb{P}(X_i = d_i)$. Let $X^{(n)} = \sum_{i=1}^n X_i$, and let s_n be the standard deviation of $X^{(n)}$, so that $s_n^2 = \mathrm{Var}(X^{(n)}) = \frac{1}{4}\sum_{i=1}^n d_i^2$. Let c and C be as guaranteed by Lemma 4.10. Let $\varepsilon = \overline{d}(A) > 0$ and choose a large n such that $\frac{|A\cap\mathcal{F}(\{1,2,\ldots,n\})|}{2^n} > \frac{\varepsilon}{2}$. By Chebychev's inequality

$$\mathbb{P}(|X^{(n)} - \mathbb{E}X^{(n)}| > ts_n) \leq \frac{1}{t^2}.$$

Hence, taking $t = 2/\sqrt{\varepsilon}$, we may choose an interval I_n of length $4s_n/\sqrt{\varepsilon}$ such that $\mathbb{P}(X^{(n)} \in I_n) > 1 - \frac{\varepsilon}{4}$. From this is follows that $B := \big\{\alpha \in \mathcal{F}\big(\{1,2,\ldots,n\}\big) : v(\alpha) \in I\big\}$ satisfies $|B| \geq 2^n(1-\frac{\varepsilon}{4})$, hence $|B\cap A| \geq 2^n\frac{\varepsilon}{4}$.

According to Lemma 4.10, the number of distinct sets $\alpha \subset \{1,2,\ldots,n\}$ such that $v(\alpha) = \sum_{i\in\alpha} d_i = T$ is at most $\frac{2^nC}{2s_n}$. It follows that $v(B\cap A) \geq \frac{2^n\varepsilon/4}{2^nC/2s_n} = \frac{\varepsilon}{2C}s_n$. From this we get $\frac{|v(A)\cap I_n|}{|I_n|} \geq \frac{\varepsilon^{3/2}}{4C}$. Letting $n\to\infty$, one deduces that $d^*\big(v(A)\big) \geq \frac{\varepsilon^{3/2}}{4C} > 0$. □

We thus come to the main result of the paper.

Corollary 4.12. *Let $(c_{ij})_{i>j}$ be an infinite, lower triangular, natural number valued matrix. Suppose that for every $j \in \mathbb{N}$, $c_{nj} = o\big(\sqrt{\frac{n}{\log n}}\big)$ as $n\to\infty$. Then $v(\alpha) = \sum_{i,j\in\mathbb{Z},\, i>j} c_{ij}$ is covering. In particular, $v(\mathcal{F})$ is a set of measurable recurrence, hence density intersective.*

Proof. Let $\alpha \in \mathcal{F}$. Then $D_\alpha v(\beta) = \sum_{n\in\beta}\big(\sum_{j\in\alpha} c_{nj}\big)$. For n large enough, one has $1 \leq \sum_{j\in\alpha} c_{nj} \leq c\sqrt{\frac{n}{\log n}}$, so by Theorem 4.11 $D_\alpha v$ is covering. The final claim follows from Corollary 3.7. □

Comparing Proposition 4.9 with Theorem 4.11, one is lead to the following.

Question. What is the precise rate of growth necessary to ensure that $v(\alpha) = \sum_{n\in\alpha} d_n$ is covering?

5 p-covering

A more general form of Theorem 3.6 will be used in this section for an application that is modestly less straightforward than Corollary 4.6.

Definition. Let $p \in \delta\mathcal{F}$ be idempotent and let $v\colon \mathcal{F} \to G$ be $\mathcal{F}$-linear. If there exists $k \in \mathbb{N}$ such that for every $A \in p$, the set $kv(A) - kv(A)$ is syndetic, then we shall say that v is *p-covering.* If $v\colon \mathcal{F} \to G$ is $\mathcal{F}$-quadratic then we shall say v is p-covering if $D_\alpha v$ is p-covering for all $\alpha \in \mathcal{F}$.

Theorem 5.1. *Let $\mathcal{H}$ be a separable Hilbert space and let G be a unitary action on $\mathcal{H}$. Suppose $p \in \delta\mathcal{F}$ is an idempotent and let $v\colon \mathcal{F} \to G$ be $\mathcal{F}$-linear or $\mathcal{F}$-quadratic and p-covering. For $f \in \mathcal{H}$ write $Pf = p\text{-}\lim_\alpha T_{v(\alpha)} f$, where the limit is taken in the weak topology. Then P is the orthogonal projection onto $\mathcal{K}_G$.*

Proof. For the linear case, note that all that is used of the premises p essential, v covering in the proof of Theorem 3.4 is that v is p-covering. The quadratic case then follows from the linear case exactly as in the proof of Theorem 3.6. □

Lemma 5.2. *There exists an (essential) idempotent ultrafilter $p \in \delta f$ having the property that for every $A \in p$ and every $n \in \mathbb{N}$, there exists $\alpha \in \mathcal{F}_n$ and $\varepsilon > 0$ such that for all $m_0 \in \mathbb{N}$ there is some $m > m_0$ with $\overline{d}(\alpha^{-1}A \cap \mathcal{F}_m) > \varepsilon 2^{-m}$.*

Proof. Let $\mathcal{L} = \{A \subset \mathcal{F} : \lim_{n\to\infty} 2^n \underline{d}(A \cap \mathcal{F}_n) = 1\}$. Then $\mathcal{L}$ is a filter and is thus contained in some ultrafilter q that is plainly a member of $\delta\mathcal{F}$. Note that for any $B \in q$, $\limsup_m 2^m \overline{d}(B \cap \mathcal{F}_m) > 0$, as otherwise $B^c \in \mathcal{L} \subset q$, a contradiction. Next pick an idempotent p of the form $p = r * q$, where $r \in \delta\mathcal{F}$. If now $A \in p$ then $\{\alpha : \alpha^{-1}A \in q\} \in r$, so that for some $\alpha \in \mathcal{F}_n$, $\alpha^{-1}A \in q$. In particular, $\limsup_m 2^m \overline{d}(\alpha^{-1}A \cap \mathcal{F}_m) > 0$, as required. □

Lemma 5.3. *Let $k \in \mathbb{N}$, $k \geq 2$, and suppose d_n is a sequence of positive integers such that $\sum_n |\frac{d_{n+1}}{d_n} - k| < \infty$. Define $v\colon \mathcal{F} \to \mathbb{Z}$ by $v(\alpha) = \sum_{n\in\alpha} d_n$. For any $\varepsilon > 0$ there exists n_0 such that for all $m \geq n \geq n_0$,*

$$|(k-1)v(\mathcal{F}_{n-1}) \cap \Phi_m| \geq (1-\varepsilon)k^{m-n},$$

where $\Phi_m = \{1, 2, \ldots, d_m - 1\}$.

Proof. First we note that the convergence of $\sum(\frac{d_{n+1}}{d_n} - k)$ is equivalent to the convergence of $\sum \log(d_{n+1}/kd_n)$, which in turn is equivalent to the convergence of d_n/k^n to some limit $c > 0$. Requiring this sum to be absolutely convergent is slightly stronger, but will hold if d_n/k^n converges rapidly enough to c. As $k > 1$ and the conclusion is unaffected by altering the first few terms d_n, we may assume without loss of generality that d_n is strictly increasing.

Let $s_m = (k-1)\sum_{i=1}^{m-1} d_i$. We aim to show that s_m is close to d_m. More specifically, define $\delta = \delta(m)$ to be the largest integer $< m$ such that

$$s_m \leq d_m + d_{m-\delta(m)}. \tag{10}$$

(We allow negative $\delta(m)$, although it is clear that $\delta(m) > 0$ for large m.) We aim to prove

$$\sum_{m=1}^{\infty} k^{-\delta(m)} < \infty.$$

As $s_m - d_m = (kd_{m-1} - d_m) + (s_{m-1} - d_{m-1})$ we have $|s_m - d_m| \leq \sum_{i=1}^{m} |kd_{i-1} - d_i|$, where for convenience we define $d_0 = 0$. Then

$$\sum_{m=1}^{\infty} \frac{|s_m - d_m|}{k^m} \leq \sum_{i=1}^{\infty} \frac{|kd_{i-1} - d_i|}{k^i} \sum_{m \geq i} \frac{1}{k^{m-i}} \leq C \sum_{i=2}^{\infty} \left|\tfrac{d_i}{kd_{i-1}} - 1\right| + O(1) < \infty,$$

where we have used the fact that $d_m \sim ck^m$. By definition of $\delta(m)$, $|s_m - d_m| \geq d_{m-1-\delta(m)}$ and hence $\sum \frac{d_{m-1-\delta(m)}}{k^m} < \infty$. As $d_m \sim ck^m$, $\sum_m k^{-\delta(m)} < \infty$.

Choose n_0 sufficiently large so that $\sum_{m \geq n_0} k^{-\delta(m)} < \varepsilon/2$ and fix $n \geq n_0$. Let N_m be the number of elements of $(k-1)v(\mathcal{F}_{n-1}) \cap \{1, 2, \ldots, d_m - 1\}$. Clearly $N_m \leq k^{m-n}$. Indeed, all elements of $(k-1)v(\mathcal{F}_{n-1}) \cap \{1, 2, \ldots, d_m - 1\}$ are of the form $\sum_{i=n}^{m-1} c_i d_i$ with $c_i \in \{0, \ldots, k-1\}$. On the other hand we shall show that

$$N_{m+1} \geq kN_m - (k-1)N_{m-\delta(m)} - N_{m+1-\delta(m+1)}. \tag{11}$$

To see this, note that the sums that are counted to get N_{m+1} include the sums counted to get N_m, plus $0, \ldots, k-1$ times d_m, provided these are distinct and less than d_{m+1}. Repeats lie in $k-1$ overlap intervals involving a previous sum. However, all such sums must be of the form $\sum_{i \leq m} c_i d_i$ where $\sum_{i<m} c_i d_i \leq s_m - d_m < d_{m-\delta(m)}$. Thus there are at most $N_{m-\delta(m)}$ repeated numbers in each overlap interval. Similarly there are at most $N_{m+1-\delta(m+1)}$ sums that are at least d_{m+1}, as all such sums can be written as $\sum_{i=n}^{m}(k-1)d_i - \sum_{i=n}^{m} c_i d_i$ with $\sum_{i=1}^{m} c_i d_i \leq s_{m+1} - d_{m+1} < d_{m+1-\delta(m+1)}$. Let $x_m = N_m / k^{m-n}$. Dividing (11) by k^{m+1} gives

$$x_{m+1} \geq x_m - (\tfrac{k-1}{k} x_{m-\delta(m)})k^{-\delta(m)} - (x_{m+1-\delta(m+1)})k^{-\delta(m+1)}.$$

As $N_m \leq k^{m-n}$, we have $x_m \leq 1$ for all $m \geq n$. Also $N_n = 1$, so $x_n = 1$. Hence for all $m \geq n$,

$$x_m \geq 1 - 2\sum_{t=n}^{m} k^{-\delta(t)} \geq 1 - \varepsilon.$$

Thus $|(k-1)v(\mathcal{F}_{n-1}) \cap \Phi_m| = N_m \geq (1-\varepsilon)k^{m-n}$ for all $m \geq n \geq n_0$. $\square$

Theorem 5.4. *Let p be as in Lemma 5.2 and let c_{ij}, $i > j$, be positive integers such that for each j, $\sum_i |\frac{c_{(i+1)j}}{c_{ij}} - j|$ converges. Let $u\colon \mathcal{F} \to \mathbb{Z}$ be defined by $u(\alpha) = \sum_{i,j\in\alpha,\, j<i} c_{ij}$. Then u is p-covering.*

We note that, in particular, any example for which for every j there is an $\epsilon > 0$ with $c_{nj} = j^n(1 + O(1/n^{1+\epsilon}))$ as $n \to \infty$ satisfies the conditions of the theorem.

Proof. Let $\alpha \in \mathcal{F}$ and put $k = \max \alpha$. Then $v(\beta) = D_\alpha u(\beta) = \sum_{n\in\beta} d_n$, where $d_n = \sum_{j\in\alpha} c_{nj}$. As c_{nj} grows as j^n for each fixed j, it is clear that $d_n/c_{nk} \to 1$ exponentially fast in n. Hence d_n satisfies the conditions of Lemma 5.3. We will show that v is p-covering. Specifically, we will show that for any $A \in p$, $(k-1)v(A) - (k-1)v(A)$ is syndetic.

Define $\Phi_n = \{1, 2, \ldots, d_n - 1\}$. Let $A \in p$. Choose by the conclusion of Lemma 5.2 $\alpha \in \mathcal{F}$ and $\varepsilon > 0$ having the property that for every m_0, there is $m > m_0$ with $\overline{d}(\alpha^{-1}A \cap \mathcal{F}_m) > \varepsilon 2^{-m}$. Pick some j_0 with $2^{-j_0} < \varepsilon$. Let $\gamma = \frac{1}{2}k^{-j_0}$. By Lemma 5.3, there exists n_0 such that for all $m \geq n \geq n_0$, $|(k-1)v(\mathcal{F}_{n-1}) \cap \Phi_m| > (1-\gamma)k^{m-n}$. We may also assume without loss of generality that d_n is strictly increasing for all $n \geq n_0$. Choose $m_0 > n_0$ with $\overline{d}(\alpha^{-1}A \cap \mathcal{F}_{m_0}) > \varepsilon 2^{-m_0}$.

For the remainder of the proof, we view, e.g., $\mathcal{F}_m$ as a subset of $\bigoplus_{i=m+1}^{\infty} \mathbb{Z}_k$. Also we use the abbreviation $\mathcal{F}_{m_0}^m = \mathcal{F}(\{m_0+1, \ldots, m\})$. Pick $m > m_0$ with

$$\left|(k-1)v(\mathcal{F}_{m_0}^m)\right| \geq \left|(k-1)v(\mathcal{F}_{m_0}) \cap \Phi_{m+1}\right| > (1-\gamma)k^{m-m_0} = k^{m-m_0} - \frac{1}{2}k^{m-m_0-j_0}$$

and

$$\left|\alpha^{-1}A \cap \mathcal{F}_{m_0}^m\right| > \varepsilon 2^{m-m_0} > 2^{m-m_0-j_0}.$$

By Theorem 4.2, one has

$$\left|(k-1)(\alpha^{-1}A) \cap (k-1)\mathcal{F}_{m_0}^m\right| > k^{m-m_0-j_0}.$$

Since $\left|(k-1)\mathcal{F}_{m_0}^m\right| = k^{m-m_0}$, one has

$$\left|(k-1)\mathcal{F}_{m_0}^m \setminus (k-1)(\alpha^{-1}A)\right| < k^{m-m_0} - k^{m-m_0-j_0}.$$

Applying v,

$$\left|v\big((k-1)\mathcal{F}_{m_0}^m \setminus (k-1)(\alpha^{-1}A)\big)\right| < k^{m-m_0} - k^{m-m_0-j_0}.$$

This implies

$$\left|v\big((k-1)\mathcal{F}_{m_0}^m\big)\right| - \left|v\big((k-1)(\alpha^{-1}A) \cap (k-1)\mathcal{F}_{m_0}^m\big)\right| < k^{m-m_0} - k^{m-m_0-j_0}.$$

We may conclude that $\left|v\big((k-1)(\alpha^{-1}A)\cap(k-1)\mathcal{F}^m_{m_0}\big)\right| > \frac{1}{2}k^{m-m_0-j_0}$. Now, for large m, $(k-1)\sum_{i\le m} d_i \le d_{m+2}$, so $\left|(k-1)v(\alpha^{-1}A)\cap\Phi_{m+2}\right| > \frac{1}{2}k^{m-m_0-j_0}$. Since m is arbitrarily large and $|\Phi_{m+2}| = d_{m+2} \sim ck^{m+2}$ for some $c>0$,

$$\overline{d}_\Phi\big((k-1)v(\alpha^{-1}A)\big) \ge \frac{1}{2c}k^{-m_0-j_0-2}.$$

In particular, $(k-1)v(\alpha^{-1}A)-(k-1)v(\alpha^{-1}A)$ is syndetic. But

$$\big((k-1)v(\alpha^{-1}A)-(k-1)v(\alpha^{-1}A)\big) \subset \big((k-1)v(A)-(k-1)v(A)\big),$$

which completes the proof. □

As in Corollary 3.7, Theorems 5.1 and 5.4 imply the following.

Corollary 5.5. *Let $(X,\mathcal{A},\mu,T)$ be invertible measure preserving and let $\mu(A)>0$. If p is as in Lemma 5.2, $\epsilon>0$ and $u(\alpha)=\sum_{i,j\in\alpha,\,j<i} c_{ij}$ where $c_{nj}=j^n(1+O(1/n^{1+\epsilon}))$ for each fixed j, then $p\text{-}\lim_\alpha \mu\big(A\cap T_{v(\alpha)}A\big)\ge\mu(A)^2$. In particular, $u(\mathcal{F})$ is a set of measurable recurrence, hence density intersective.* □

Question. Let $(d_n)_{n=1}^\infty$ be a sequence of natural numbers and assume there is a $k>0$ such that $d_{n+1}\le kd_n$ for all $n\in\mathbb{N}$. Must $v(\alpha)=\sum_{n\in\alpha} d_n$ be covering? If not, must v be p-covering for p as in Lemma 5.2?

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