Monochromatic permutation quadruples–a Schur thing in S_n

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Abstract

Schur proved that for any finite partition of the naturals, some cell contains two numbers and their sum. We use Ramsey's theorem to prove a noncommutative Schur theorem for permutation quadruples $\{x, y, xy, yx\}$.

By Schur's theorem [2], for any $r \in \mathbf{N}$, there exists $n \in \mathbf{N}$ such that for any r-coloring of $\{1, 2, \ldots, n\}$, one has a monochromatic "Schur triple." That is, a 3-element set of the form $\{x, y, x + y\}$. The simplest proof is via Ramsey's theorem [3], which states that for every $k, n, r \in \mathbf{N}$, if the k-member subsets of a sufficiently large set B are colored in r colors there is an n-member set $A \subset B$ whose k-member subsets form a monochromatic family.

Indeed, from $\{1, 2, \ldots, n\} = \bigcup_{i=1}^{r} C_i$ one may induce an r-coloring of the 2-subsets of $\{1, 2, \ldots, n\}$ by the rule $\{i, j\} \in D_t$, $i < j$, if $(j - i) \in C_t$. For large enough n one may choose by Ramsey's theorem a set $\{a, b, c\}$, $a < b < c$, with $\big\{\{a, b\}, \{b, c\}, \{a, c\}\big\}$ monochromatic for the induced coloring, so that $\{b - a, c - b, c - a\}$ is monochromatic for the original coloring. Let now $x = b - a$ and $y = c - b$. Then $x + y = c - a$, so we are done.

In a general group G, if $xy \neq yx$ then the 4-member set $\{x, y, xy, yx\}$ forms what might be called a "Schur quadruple." There can be no partition theorem for Schur quadruples true of abelian groups, which contain none, while for free groups the matter remains open. It is therefore natural to ask if there are groups for which a version of Schur's theorem for quadruples holds. The answer is *yes*; Ramsey's theorem provides a familiar proof.

Theorem. Let $r \in \mathbf{N}$. There exists $n = n(r)$ such that for any r-coloring of the symmetric group S_n, there is a monochromatic Schur quadruple.

Proof. Choose by Ramsey's theorem an n such that for any r-coloring of the 3-element subsets of $\{1, 2, \ldots, n\}$, there is a 4-element set whose 3-element subsets comprise a monochromatic family. Now suppose we are given an r-coloring $S_n = \bigcup_{i=1}^{r} C_i$. Put $\{i, j, k\} \in D_m$ if and only if $(ijk) \in C_m$, where $i < j < k$. By choice of n, there is a 4-element set $\{a, b, c, d\}$, with $a < b < c < d$, all of whose 3-elements subsets lie in the same cell D_y. This gives $\{(abc), (acd), (abd), (bcd)\} \subset C_y$. Observe now that $(abc)(acd) = (abd)$ and $(acd)(abc) = (bcd)$. $\square$

It was recently shown that such a Schur theorem for quadruples holds for a broad class of sufficiently noncommutative *amenable* groups (see [1]).

References

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3. F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* Series 2 **30** (1930) 264-286.

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